University of Alberta

Being in a Mathematical Place:
Brief Immersions in Pure Mathematics Investigation

by

David Wagner ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial
fulfillment of the requirements for the degree of Master of Education.

Department of Secondary Education

Edmonton, Alberta
Spring 2002

National Library
of Canada

Acquisitions and
Bibliographic Services

395 Wellington Street
Ottawa ON K1A 0N4
Canada

Bibliothèque nationale
du Canada

Acquisitions et
services bibliographiques

395, rue Wellington
Ottawa ON K1A 0N4
Canada

Your file Votre référence

Our file Notre référence

The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

L'auteur conserve la propriété du droit d'auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

0-612-69669-3

Canada

University of Alberta

Library Release Form

Name of Author: David Wagner

Title of Thesis: Being in a Mathematical Place:
Brief Immersions in Pure Mathematics Investigation

Degree: Master of Education

Year this Degree Granted: 2002

Permission is hereby granted to the University of Alberta Library to reproduce single copies of this thesis and to lend or sell such copies for private, scholarly or scientific research purposes only.

The author reserves all other publication and other rights in association with the copyright in the thesis, and except as herein before provided, neither the thesis nor any substantial portion thereof may be printed or otherwise reproduced in any material form whatever without the author's prior written permission.

3603 – 109B Street
Edmonton, AB T6J 1C9

Date: Dec. 21, 2001

University of Alberta

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled *Being in a Mathematical Place: Brief Immersions in Pure Mathematics Investigation* submitted by David Wagner in partial fulfillment of the requirements for the degree of Master of Education.

David Pimm

Dr. David Pimm
(Supervisor)

Dr. Lynn Gordon Calvert

Dr. Elaine Simmt

Date: Dec 19/01

Abstract

In this study, two Alberta Pure Mathematics 10 classes engaged in investigative pure mathematics projects for the first time. The disorientation demonstrated by both the teachers and the students is compared with the experience of unfamiliarity characteristic of sudden immersion in foreign places. Interpretation is informed by studies of the more formalized and regularly used mathematical investigations in Britain.

Selections of transcripted dialogues and students' written work are used to identify the participants' tensions in the exploration environment. As the teachers aimed to initiate helpful interventions they struggled with how to position themselves with regard to guiding their students.

The students were captivated by the problems in these mathematical landscapes and were not satisfied with simplistic responses to the complexities they found. Instead, many of them were propelled into zones of creativity opened up by the rejection of simplicity. Their writing, with its variety, demonstrates different interpretations of implicit values.

Acknowledgment

I am thankful for the patience and expert support I have experienced from many good people in this research endeavour. In this regard, I want to especially recognize my advisor, Dr. David Pimm and my wife, Carolyn Wagner.

I am thankful in a different way for the willingness of my research participants to expose themselves to new experiences and to my involvement in their classes. Both participant teachers clearly demonstrated a commitment to the wellness and growth of their students. I believe that the extent to which their students were open to my participation attests to the good relationships nurtured by these teachers.

Others that deserve thanks include the many people who have contributed to the fullness in my life. Relationships with these people have opened up for me new perspectives from which to view the world around me. I think particularly of my children, Ruth and Sophia Wagner, my parents, Kurt and Anne Wagner, my parents-in-law, Donald and Kathleen Neufeld and my adopted Swazi parents, the late Elijah and Mavis Simelane.

Table of Contents

1. **Setting** **1**

    Significance 3
    Research Question 4
    Contextual Limitations 4
    Definitions 5
    The Issue of Time 7
    Outline 9

2. **A Tree in a Field: Reviewing the Literature Surrounding the *Investigation*** ... **11**

    Building the Tree House – Teachers Coexisting with Investigation 13
        Formative Voices 14
        Voices in Dialogue 16
    Stepping out of the Tree – Critical Inward Looking 18
    A View from the Tree – New Insights into Mathematics Education 20
    Blowing Leaves – Investigation Outside of the United Kingdom 23
        Considering the Classroom Environment 24
        Considering the Nature of Mathematical Problems 26
    A Settling Seed 29

3. **Planting a Seed of Hope: Method** **30**

    Preparing the Ground – Selection of Mathematics Classes 32
    Seeds of Growth – The Investigative Projects 33
    Watching for Growth – Methods of Observation 38
    Gentle Gardening – Ethical Considerations 41
    Harvest 42

4. **Ways of Seeing a Mathematical Place** **44**

    Ways of Seeing 46
    A Mathematical Place 47
    Being in a Foreign Place 49
    Ways of Being 50
    Problems – What is Real in Mathematics? 52
    Ways of Guiding 54
    Modelling Different Ways of Being 56
    A Basis for Interpretation 62

5. Two Instances of Brief Immersion in a Mathematical Place 63

- Scene One – Mr. Penner and Natalie 63
  - Mr. Penner's Way of Guiding 68
  - Natalie's Way of Being 80
- Scene Two – Greg and Mrs. Foster 86
  - Mrs. Foster's Way of Guiding 87
  - Greg's Way of Being 89
  - Responses to Greg's Frustrations 94
- Summary 98

6. Some Characteristics of Brief Immersion 99

- Captivation 100
- Complexity 104
- Student Writing 107
- Creativity 118
- Summary 133

7. Reflections 135

- Summary 135
- That Which I Had Hoped to See 138
- Tensions 141
- Some Uncovered Problems 143
  - The Problem of Real Problems 144
  - Giving Problems 145
  - Problematic Discourse 147

References 149

Appendix 1: Scoring Rubrics 155

Appendix 2: Letters of Consent 158

List of Figures

Figure 1. Timeline for class participation 34
Figure 2. The investigative projects 36
Figure 3. The multidimensional place 48
Figure 4. Outfitted immersion 56
Figure 5. Neighbourly immersion 58
Figure 6. Transmission model (tourism) 59
Figure 7. Goldsmith and Shifter's (1997) models of classroom dynamics 60
Figure 8. A representation of Jaworski's (1994) teaching triad 61
Figure 9. My way of seeing "orchestration" 61
Figure 10. Natalie, Kathy and Teresa's report on "Playing with Squares" 68
Figure 11. An explanation of Natalie's ratio-based trajectory 69
Figure 12. Mr. Penner's ways of guiding in one instance 74
Figure 13. Time use by Natalie, Teresa and Kathy 83
Figure 14. Calculating the quotients in "Parallel Division" 91
Figure 15. Testing the parallelism conjecture 92
Figure 16. "Parallel Division" work by Greg, Michelle and Angela 97
Figure 17. Some student reflections on valued mathematical text 109
Figure 18. "Parallel Division" work by Aaron, Brent and Don 110
Figure 19. "Parallel Division" work by Glen, Wes, James and Jason 112
Figure 20. "Playing with Squares" work by Shawn, Terry and Brian 116
Figure 21. An expected response to "Playing with Squares" 120
Figure 22. Gordon's stacks of squares 122
Figure 23. Modeling $\sqrt{72} = \sqrt{8} + 4\sqrt{2}$ 123
Figure 24. "Playing with Squares" work by Jennifer, Kara, Tasha and Chantelle 124
Figure 25. Figures at which Tasha might have been pointing 126
Figure 26. "Playing with Squares" work by Jon, Aaron and Brent 127
Figure 27. "Playing with Squares" work by four ESL Students 129
Figure 28. Student conjectures regarding "Parallel Division" 131

Chapter 1 – Setting

Over the last few years, mathematics curriculum change in Alberta secondary schools has captured the attention of teachers, students, school administrators, parents, universities and the press (e.g. see McCabe, 2000, pp. 110-118). Reactions range from creative excitement to vocal disapproval. Teachers bear the brunt of criticisms from parents at the same time as they are encouraged to embrace the philosophical and content shifts in the new program. A shift toward more project-based learning and assessment is largely responsible for some of the more extreme reactions. In this study, I reflect on the student and teacher reactions and responses to my introduction of pure mathematics projects into two classes unfamiliar with open-ended mathematical exploration.

Two formative experiences have inspired my interest in investigating this curriculum shift. Firstly, an increased awareness of culture and its interrelationship with curriculum grew from my three years living and teaching school mathematics in a small town in Swaziland, Africa. Immediately following this international experience, I was a lead teacher with the responsibility to lead my school district's mathematics teachers through their struggles with the new Alberta high school program of studies in mathematics. Partly because of my intercultural experiences, I was acutely aware of certain aspects of Canadian culture's shaping force in this program of studies, both in my colleagues and in myself. I wondered how the curriculum change would in turn shape culture both within the classroom and beyond.

Within the emerging Alberta mathematics curriculum, a new cluster of senior high school courses has been designed for students not aiming for studies in university sciences. This set of three courses for students in grades 10 through 12 is called Applied

Mathematics. These courses exemplify a new investigative approach with resources structured around context-based projects that welcome a diversity of approaches. By contrast, the new Pure Mathematics courses, which form the strand designed for students destined for further study in mathematics and the sciences, carry only minor philosophical changes from their predecessors, Mathematics 10, 20 and 30. Although the approved textbooks include some language that suggests investigation and exploration, students are generally drawn along very particular paths to particular conclusions. As I understand the terms, *exploration* and *investigation* in mathematics are used to avoid foregone conclusions and embrace a multiplicity of approaches, though not necessarily a diversity of solutions.

I sensed that while some teachers were willing to try new investigative approaches with students who would not require a strong mathematics background for university, few were willing to adjust their processes of teaching for their university-bound students. From 1998 to 2000, Alberta's provincial assessment of the matriculation-bound mathematics students was led by McCabe, who is presently the high school mathematics consultant for the province's second largest school district. Under McCabe's leadership, the province's department of education began the implementation of an annual project for each of the strands' grade twelve students, one for Applied Mathematics and one for Pure Mathematics. While these department-developed projects were not mandatory, they formed the basis of one question on each diploma exam. All Pure Mathematics 30 and Applied Mathematics 30 students in the province write the summative diploma exam, which is set by the provincial assessment department. The project-based question would be worth ten percent of the exam mark, and thus five percent of each student's cumulative course mark. It seems to me that teachers understand these projects to be virtually mandatory, because of the heavily-weighted exam question that relates to them. Because of her philosophy of pedagogy, and also to support the preparation of students for their grade 12 projects, McCabe encourages the idea of project work for students in the grades 10 and 11 Pure Mathematics courses (personal communication, October 4, 2000).

Partly because of her encouragement, I studied two particular high school Pure Mathematics classrooms in which students and teachers were to contend with pure mathematics projects that I developed. These projects would call for creative solutions based on both individual and group exploration. Before actually collecting data, I described my two primary objectives in this way:

1. examine within the classroom culture the shifts that accompany project work;
2. use the cultural shifts within the classroom as a source of insight into possible implications of investigative projects.

Significance

Most recently, scholars interested in mathematics education's cultural implications (e.g. Bishop, 1994; Skovsmose and Nielsen, 1996) have been calling for study of the enculturation in particular classroom cases. For instance, Bishop (1994) suggests that research "be focussed on the cultural framing, and hidden assumptions involved in classroom activities" (p. 18). Although I felt called to this research endeavour by these scholars and their writing, my principal aspiration was and is to contribute to the educational experience of teachers and students.

Research Question

I stated my primary research question in this way, "How does classroom culture change when project work is introduced into a mathematics class?" Although I planned to approach my chosen settings with an interest in cultural change, I expected that more specific issues would emerge in these particular places and times. While I brought a number of particular questions into the study – questions that would draw my attention during observation – I expected that other questions would form as I participated in these classroom settings. I had eight questions in my original list:

1. What evidence exists of instructions in this project work being unfamiliar to these classroom participants?
2. How do these *students* respond to instructions that seem unfamiliar to them?
3. How do these *teachers* respond to instructions that seem unfamiliar to them?
4. How do these *teachers* try to influence their *students'* interpretation of the unfamiliar parts of the instructions?
5. How do these *students* try to influence their *teachers'* interpretation of the unfamiliar parts of the instructions?
6. In what ways do the unfamiliar parts of the instructions seem to free these *students* to think and act in new ways?
7. In what ways do the unfamiliar parts of the instructions seem to free these *teachers* to think and act in new ways?
8. What do these classroom participants say about the value of the unfamiliar parts of the instructions?

Contextual Limitations

I was aware that the questions I brought into this study would only allow a glimpse of possible implications of open-ended projects in mathematics classrooms. For the sake of careful observation, I limited my study to two particular classes, both observed over short time spans. Within these two contexts, I expected to feel confident about my analysis of changes in these particular places during the time of my participation. While I cautioned myself that any predictions that I would make would be speculative, I hoped that evident changes in students' or teachers' thinking about and doing mathematics during this time could help provide insight into possible implications such project work might have on the future of the participants of these classes.

As I expected, my immersion in these two particular classroom settings changed my view of what is important to share with other teachers and mathematics education scholars. In retrospect, I can see that before my data collection I felt that valuable study must aim to address the future. I worked hard, especially in interviews with participant students and teachers, to uncover evidence of possible changes in the way they would do things subsequent to the project-work experience.

Instead, I found myself captivated by the present instead of the future. I became more interested in how my participants thought and felt during their project work than in how the projects might divert their trajectories for future mathematics and future approaches to problems. Most of my analysis here thus concentrates on describing ways of seeing and ways of being, rather than on divining implications and prescribing strategies to produce future desirable effects.

Definitions

My research interest in classroom culture required a clear understanding of how I define the word *culture*. The word can carry many meanings and is important both to my original research expectations and to my retrospective interpretation. The definition that stands out for me is one I heard from a Mohawk elder who said that the way he and his people see culture is that it is simply "the things we do". This way of seeing culture makes it highly observable. The elder's definition fits well with more formal definitions that focus on the products of human work and thought. For example, the Oxford English Dictionary points to "the civilization, customs, artistic achievements, etc., of a people, esp. at a certain stage of its development or history" (OED, 1989, "culture *n*.").

Such formal definitions and the context of the elder's comment focus on the influence of a people's heritage on its activity and its products. These influences are not easily observed. In this thesis, I think of culture as the things the people in a particular social grouping and setting do under the influence of their collective history. Mathematics classrooms can be seen as micro-societies that carry their own culture while being influenced by their places in wider educational and community cultures.

Students and teachers in any class develop a unique micro-culture. Many expectations and routine procedures become implicit. This micro-culture is influenced by other larger cultures – larger social bodies that carry their own expectations and routine procedures. The community of mathematicians, the mathematics educators' community, the local school community, the community in which the school structure stands, the provincial and national communities, and even the current global community influence the participants in the class.

With this view of culture in mind, I am interested in observing the participants in each classroom setting as they are confronted by new ways of doing mathematical tasks – tasks that differ in nature from their particular routine tasks. These tasks are not necessarily new to the world; neither do they contain new mathematical content.

The projects I developed for use in this research and elsewhere are explicitly pure mathematical explorations. The investigations are not set in a story. Although I take an active interest in peoples' stories and their very real problems in this world, I also value the study of pure mathematics in its own right. Pure mathematics, as I see it, is the study of relationships and rules of procedure abstracted from the context in reality. It can provide a wonderful environment for developing a facility for creative thinking about perspective and procedure. People who are creative in this way can make invaluable contributions to the people in their world. My understanding of the value of pure mathematics undoubtedly influences the way I perceive students and teachers involved in its discourse and activity.

The Issue of Time

This presentation of my research experiences and interpretation is structured to follow a traditional thesis format. The format implies to me a linear progression and unfolding of time. While I recognize that time is, in fact, linear, I find writing about its progression problematic. Although I know I have a past, it is relatively inaccessible to me – not completely inaccessible, but definitely less accessible than it was when I was living it. There are things I think I know about my past, but my memory is not a reliable record, nor is it publicly accessible. Even aids to memory, like, for example, text I have written, can only be interpreted by the person I am now or by other people as they exist in the present.

A review of literature might seem more static than a recollection of lived experience. The present tense may seem like a reasonable choice for presenting a review because the words in the literature are the same now as they were before I conceived of my research problem. But, since I wrote my literature review after having collected my data, it might seem more appropriate to place the review later in the thesis or to use verb tenses to place it chronologically later than my presentation of data.

Although I wrote my literature review after collecting the data, I first encountered some of the literature before, some after and some both before and after my data collection. Surprisingly to me, the literature I read both before and after was different for me before my research experience from after. The words seem to have changed as I recalled them meaning one thing at one time and something else at another. Perhaps this phenomenon is better understood when I see myself *making* meaning of the text rather than *finding* a fixed meaning in it. My new experiences have changed and continue to change the way I interpret the past.

It is the same for the analysis of my data. It seems now that the events in the classrooms meant one thing to me when I was present with their unfolding and mean something else, or perhaps mean more now at the time of writing. As with my understanding of textual interpretation, I prefer to see myself making meaning of the data rather than finding meaning in it.

These time and perception problems have influenced my choices for structuring this thesis in two significant ways of which I am aware. I have chosen to write the bulk of the thesis in the present tense. When I refer the reader forward or backward, I use the present tense to describe what is there. For example, in chapter 5 I might write, "In Chapter 3, I suggest … and, in Chapter 6, I discuss…". Chapter 3 and Chapter 6 are present when Chapter 5 is being read, even though, for the reader, Chapter 3 has passed and Chapter 6 is to come. I use past tense when I attempt to reconstruct particular thoughts and experiences. Some of these reconstructions are nearly word-for-word from my earlier writing, and others rely more heavily on memory. In either case, I am reminded that the past has in fact passed. It is like a deceased friend, forever with me in memory and also lost forever.

Chapter 4, in which I describe my way of seeing – my way of interpreting the data – is inserted into the traditional thesis structure because of my understanding of the dominance of the present in my interpretation. After reflecting on my experiences of the students and teachers who participated in the research, I saw the data differently from the way I did when participating in the classroom events. The way I see the events now is the one that dominates my way of interpreting the data; thus, I include a chapter on this way of seeing before reporting on what happened in the classrooms.

Outline

Although the inferred temporal linearity inherent to the traditional thesis format seems peculiar to me, I am content to follow it because it seems to me that any alternative linear format would suffer the same complications. This format also offers intriguing parallels to the human lifespan. The first three chapters describe my recollection of the birth of my investigation. In Chapter 1, I describe its conception. I offer my sense of how my prior experiences influenced me to be interested in this topic of study, and I describe my expectations for the study that I wished to birth. In the literature review, I observe other active teachers and researchers who have taken part in birthing elder cousins to my interest in pure mathematics investigative projects. Chapter 3 describes my plan for delivery – how I planned to bring my idea to life.

In Chapter 4, I describe my way of seeing that which grew out of my plans. In Chapter 5, I interpret two particular scenes from these experiences. In Chapter 6, I look at this emergent life in a different way, using a diversity of interactions from these classrooms to demonstrate themes that seemed to emerge in my interpretation.

Even as our children grow, we begin to see the legacy they will leave, the impact they make in the places they live and the possibilities they open up but do not yet address. In my final chapter, I reflect on my emerging understanding of the place of this research in its world. I ask what it has done and what it leaves undone.

Chapter 2 – A Tree in a Field: Reviewing the Literature Surrounding the *Investigation*

I watched my toddling daughter play with other children in a field beside the church hall at a recent wedding. She was captivated by one tree. For her, this tree was the essence of the field. She hardly seemed to notice the other trees or children or bits of enticing garbage strewn about, which undoubtedly could be seen as equally marvellous through the eyes of another. I cannot be sure why this one tree consumed her attention, for my attention was captivated by her, one child among many children playing in this field. My description of this field focuses on observations about this one person, who in turn focused her attention on one tree.

For me, a review of literature related to my research study is like my daughter's infatuation with one particular tree in a field of many delights. Here, I provide my perspective on one "tree", and interpret the field surrounding it in terms of this one living thing. My description of the tree is limited not only by my perspective but also because it is a living thing, changing, growing and dying all at once in every moment. The living and dynamic thing that has consumed my attention while reviewing literature that relates to my research is the mathematical investigation, especially in its form found in the United Kingdom.

While mathematical investigations were already a part of existing intermittent practice, the *institutionalization* of the mathematical investigation as a particular and important thing in the school mathematical culture of the United Kingdom is often attributed to the publication Mathematics Counts: Report of the National Committee of Inquiry into the Teaching and Learning of Mathematics in Schools (DES, 1982) known as the Cockcroft report, which lists elements of good mathematics teaching. This list calls for the inclusion of opportunities for "investigational work" (p. 71) distinct from "problem solving", which encompasses "the application of mathematics to everyday situations" (p. 71). Investigations involved exploration of pure mathematics (Morgan, 1998, p. 56). Jaworski (1994) conducted a deep investigation into this relatively new thing called an investigation. She describes it like this:

> Mathematical investigation seemed to involve students in loosely-defined problems, asking their own questions, following their own interests and inclinations, setting their own goals, doing their own mathematics and, moreover, having fun. (p. 3)

Morgan (1998) suggests that an investigation has the following features (p. 59):

- it is essentially mathematical;
- it relates to patterns, relationships, generalizations;
- its learning objectives value process rather than content;
- it is exploratory and creative, and may have multiple valid outcomes;
- it is part of good classroom practice – hence it ought to be assessed.

She reports that curriculum documents, teacher journals and books giving advice regarding investigations all agree that investigational work is real mathematics, is open, creative and empowering, and should permeate the curriculum (p. 72).

Following the Cockcroft report, *coursework* formed part of the summative high-stakes assessment for high school mathematics. Coursework involves extended school-based project work written up by students (see Morgan, 1998, pp. 37 and 56). As one might expect, teachers would implement "investigations" according to their own or their mathematics department's interpretation of *investigation*.

Criticism of the word soon followed. For instance, the Association of Teachers of Mathematics' journal Mathematics Teaching provides a rich source of insight into the dialogue surrounding "investigation" and the history of its development. Delaney (1986), a primary school teacher, uses this platform to ask what an investigation is. He notes, "the word 'investigation' existed before Cockcroft but as a consequence of being used in that report has become a different word with a different range of meanings" (p. 16). He complains that the use of the word *investigation* "focuses on some abstract activity that happens without people" (p. 16). He recalls the intent behind the Cockcroft report – to get students involved in mathematical thinking.

Building the Tree House – Teachers Coexisting with Investigation

No matter how one defines or likes the word *investigation*, it is a word that teachers in the United Kingdom have had to live with since the Cockcroft report. In this part of my review of the literature, I look at exemplars of teachers' interaction with students in this new environment that includes investigation, and at literature that seems to be formative for these teachers. If we imagine this particular environment as a tree within the larger field of writing about mathematics pedagogy, then we might think of the educators talking among themselves about how to live in that tree. Their discourse builds for them a house to improve their experience of living in the tree.

Formative Voices

Mason, based for the last thirty years at the Open University (an adult distance teaching institution) in the United Kingdom, has had a special interest in teaching investigatively since his entry into the mathematics education environment. Many mathematics teachers took mathematics courses shaped by him – courses which engage participants in mathematical investigation (J. Mason, personal communication, May 23, 2001). Because of their engagement in mathematical exploration, these teachers, who came from all over the United Kingdom, would be able to sympathize with the experiences of their students involved in investigations.

Mason, Burton and Stacey's (1982) book Thinking Mathematically might be just as important. It is an atypical mathematics book because it has no answers to the many problems it poses. This structural feature is a clear departure from the common idea that there is a particular or optimal answer to every mathematical question.

In the first half of the book, they thread through a succession of problems suggestions about possibilities for working on mathematical problems in general. While trying to avoid normalization of any particular process, they provide insight into possible ways of thinking mathematically. Recurring phrases accompanying suggested strategies include "Stuck?" and "Try it now" (Mason *et al*., 1982, p. 6, for example). In their advice, they introduce general problem-approaching sequences: entry-attack-review (p. 31) and check-reflect-extend for the review phase (p. 43). They place conjecturing at the centre of mathematical thinking (p. 75), and problematize the idea of justification. They assert that one "can rarely be absolutely sure" (p. 105) that a conjecture is convincingly justified.

After walking their readers through various strategies for approaching problems, Mason *et al.* (1982) stress the need for mathematical thinkers to free themselves from reliance on external hints and suggestions. The remainder of their work attempts to free the budding thinker from dependence on such external helps. Teachers who engage in mathematics under the guidance of Mason *et al.* ought to be able to transfer their experiences into their classrooms, and set their students free to think mathematically.

More recently, Mason and Watson (1998) have compiled a guide for educators interested in posing promising problems for mathematical thinking. They provide exemplars ranging from primary to undergraduate level for a spectrum of mathematical structures. Within each of these categories, their exemplars cover an array of mathematical processes, including: exemplifying and specializing, completing and correcting, comparing and sorting, changing and reversing, generalizing and conjecturing, and explaining and justifying.

Others who seem to have been formative in teachers' use of mathematical investigations include Wheeler and other editors of Mathematics Teaching. Wheeler (1988) suggests that mathematical investigations are like crossword puzzles – the experience of the process is often more important than the end-point (p. 303). Thus, he advises teachers to be careful about intervention when managing investigations (p. 305) in favour of allowing students to experiment (p. 304). These suggestions repeat his earlier contribution (Wheeler, 1984) in Mathematics Teaching's regular collection of short contributions in the ongoing column entitled "Gatherings". There, he challenges the idea that teachers can help students with general problem-solving strategies: "to try to teach systematic thinking is probably a mistake, even if we knew how" (p. 25). Although he seems to contradict the idea behind the guide to thinking mathematically introduced by Mason *et al.* (1982), I suggest that his criticism is focused against time-saving strategies that speed up student responses to problems. The work of Mason *et al.* seems to stretch the time by stressing process, critical thinking and extension.

Fielker, during his term as editor of Mathematics Teaching, was instrumental in the ongoing dialogue in at least two ways. He promoted mathematical investigation in his written contributions which include both strongly-worded criticism of the transmission approach to pedagogy (e.g. Fielker, 1982) and constructive suggestions for investigative settings (e.g. Fielker, 1983). He was also influential by providing forums in the journal for dialogue surrounding issues of mathematics education – providing space in the regular "Gatherings" for discourse such as Wheeler's, and in the regular "Passages" in which teachers inspired each other by sharing excellent student work from investigative settings.

Tahta and Hemmings were the co-editors who started the "Gatherings" and "Passages" features. "Passages" was a part of the journal from 1983 until 1992, and "Gatherings" from 1983 until 1986.

Voices in Dialogue

Hewitt's (1983) contribution typifies the kind of sharing that Mathematics Teaching's "Passages" embodies. He describes an investigative discussion that ended in his marvelling over a particular student's work:

> I had intended to end there but in the following lesson Debbie said she had a way of working out the answer. ... The rest of the lesson was spent with the whole class including myself trying to understand her method and

> testing it out with lots of different examples. They all worked – much to our surprise. (p. 14)

In her own article, the amazing work of Debbie Frankham (1983), the student, follows Hewitt's report of the experience.

The selection of articles in Mathematics Teaching also attests to the importance of investigative settings in United Kingdom mathematics classrooms. Baker (1986), for example, responds to Jaworski's (1985) earlier description of a class experience with a poster called "The Great Dodecahedron". Baker describes, from her interaction with her students, a very different investigative path inspired by the poster.

Teachers use this forum to describe difficulties as well as successes. Shuller (1983), for example, comments on her experience with students using geometry films saying, "I saw how difficult it was to be 'with' each student" (p. 38). Cornelius (1985) points to another source of difficulty with his collection of fictional letters from parents concerned with curriculum innovations: "I am concerned that [my son] will not know any real arithmetic, algebra, etc." (p. 38). Edmonds (1983) agrees, saying innovation is not easy because of parental resistance and pressures from "students used to being dependent on the teacher" (p. 33).

Mason (1988), well aware of the struggles facing innovative teachers, expresses encouragement for them. "Most tensions are endemic and inescapable. Getting them out into the open means that they can be robbed of their numbing effect, and turned instead into potent sources of energy" (p. 164). He discusses a cluster of tensions associated with investigative learning environments: control, time, confidence versus challenge, product versus process, autonomy, intervention. Watson (1986), another leader in an investigative approach to teaching and learning mathematics, is as encouraging in her own way.

"Strangely enough, I think self-doubt is an ideal state of mind. … The teacher who knows all the answers and expects students to produce a fixed sequence of arguments leading to some final conclusion is not the best person to draw creative thought from a class" (p. 16). Mulholland (1985) recognizes, from her position as mathematics department head, that above all else it takes *time* for teachers to become fluent with a new approach to teaching mathematics.

Stepping out of the Tree – Critical Inward Looking

The dialogue I have cited so far can be characterized by the question "How can we best conduct mathematical investigation?" The discourse also includes critical thinking – questioning typical investigation practices and asking again what investigation is. Such critical thinking might be compared with a situation where a group of children is building a tree house and one child climbs down the tree to analyze their progress from the outside.

One recurring theme in the criticism is the concern that, despite an interest in open-ended investigation, there seems to be a tendency for teachers to establish a too-specific idea about what good open-ended work looks like. Such particular expectations implicitly close the open-ended nature of good investigations. Love (1988) declares that *process* ought to be a verb, not a noun because "the 'processes' are becoming additional 'content'" (p. 257) in too many classrooms. He worries that the unorthodox strategies characteristic of good problem solvers will be trained out of them.

Hewitt's (1992) famous criticism provides a particular example of the kind of problem Love warns against. Hewitt is saddened by the fixation on generating tables in investigations. He claims that making general statements derived from tables is to make "statements about results rather than the mathematical situation from which they came" (p. 7). He likens such activity to train spotting, an activity in which the hobbyist searches for trains but ends up collecting mere numbers.

The most exhaustive study of investigative teaching also shares Love's concerns. In her ethnographic study of grade eight classes in which teachers sought to teach investigatively, Jaworski (1994) develops a philosophy rooted in constructivism, an alternative to the popular belief in teaching-as-transmission. Although she expresses concern for her biases (p. 69), she recognizes from her research that "belief in some form of transmission process made teaching easier or more bearable for the teacher" (p. 84). Like Mason (1988) and Watson (1986), Jaworski recognizes and embraces tensions inherent in investigative settings.

Among the teachers with whom Jaworski (1994) worked, perhaps the most prominent tension centred on control. She provides a journal excerpt from one of these teachers who describes the tension well: "I'm controlling [the] direction because I think that people should have freedom – which is a complete contradiction" (p. 150). *Requiring* students to complete an investigation project and *expecting* them to embrace its open-endedness manifest teacher control at the same time as inviting freedom.

Like Mason (1988) predicts, out of this tension Jaworski (1994) finds potent approaches for teachers working in investigative settings. Teachers can focus on the classroom ethos, teaching students how to learn from each other and support each other (p. 177). With regard to mathematical conventions, Jaworksi describes how her exemplar teachers promoted "established meanings while valuing students' individual perceptions" (p. 176).

Jaworski's (1994) questions and observations surrounding the implementation of new approaches are especially important for teachers interested in using investigations. She reminds teachers interested in beginning such work that "the implementation of investigative work is not just a matter of *doing* things differently, it also involves a different way of thinking" (p. 184). She claims that sound basic philosophy together with a sensitivity to classroom mathematical culture are necessary for successful implementation. With these basics in place, as the teacher begins to construct an investigative classroom ethos he or she needs to be afforded the opportunity to "step back from the event to try to see it less subjectively in order to examine it critically" (p. 192). Jaworski concludes with this observation:

> The view of learning which I have come to value is one in which individual constructions are influenced by cultural domains and social interactions, and the social and cultural environments are continually regenerated by actively cognizing individuals. (p. 212)

A View from the Tree – New Insights into Mathematics Education

Imagining the tree in the field again, I see that the children up in the tree have a different view of the field than children in other trees or on the ground. Similarly, mathematics educators who have immersed themselves in the culture that involves mathematical investigations have been afforded a unique view of the field of mathematics education. In this section of my review, I focus on literature that speaks to mathematics education more generally – literature that is informed by experiences with investigative settings. I believe that it is significant that the more general insights in this part of the literature come from the people who were active in critical inward looking.

Jaworski (1994) exemplifies the move from critical inward looking toward more general insight. Her work might fit into any of my first three categories of literature. Her work moves beyond a description and analysis of what was happening in investigative classrooms, and toward the development of theory for teachers interested in applying the constructivist philosophy to their practice.

Focusing her gaze on a particular part of the discourse, Ainley (1988) develops a taxonomy of teacher questions that provides insight into the nature of classroom relationships. Her work follows a thread begun in Mathematics Teaching by Smith (1986) who weighs the possibilities created when a teacher only asks questions and tells students nothing in their investigative work. Ainley (1987) responds with concerns about questions, saying that teachers' questions are rarely genuine. Teachers' "guess-what's-in-my-mind" questions create the "illusion of participation" (p. 24). Andrews (1987) continues the dialogue, suggesting that, if Ainley were to be taken seriously, then the whole "school game" would have to be questioned. It became clear that Ainley's interest had been piqued by the exchange when she introduced her taxonomy of questions at an International Psychology of Mathematics Education Conference (Ainley, 1988).

Hewitt (1987) became captivated by yet another aspect of learning that is exposed when students are engaged in investigations. He describes for the Mathematics Teaching audience the insight he has gained into memory through his interaction with students indulging in underlying problems in an investigative setting. Here he describes the *arbitrary* as terminology and mathematical conventions. Hewitt suggests that if the "arbitrary is used as a means to work further at the problem" (p. 19), attention is focused on the problem and the arbitrary is easily remembered. To memorize arbitrary rules, on the other hand, is difficult.

He continues this vein of thinking in a recent series of articles. Firstly, Hewitt (1999) discusses the differences between the *arbitrary* and the *necessary*, and suggests appropriate teacher strategies for teaching each of these kinds of knowledge. The arbitrary is described as things that *cannot* be worked out and *might* be so, whereas the necessary *can* be worked out and *must* be so. In the second of his series, Hewitt (2001a) suggests that the arbitrary ought to be subordinated to the necessary to assist memory. He closes the series with suggestions for educating awareness (Hewitt, 2001b).

Morgan (1998), stimulated by teachers' recognition that students' good work was not adequately represented in their written responses to open-ended tasks, has investigated the writing up of mathematics. Her research is based on interviews with teachers who were evaluating student investigation reports external to the schools where they were written.

She distinguishes between various domains of mathematical discourse including research, inquiry, journals, and schools (Morgan, 1998, p. 11), and focuses on student writing. Regarding the claims made by proponents of "writing-to-learn", she counters that their claims are inadequate because of their "assumption that the interpretation of writing produced by students is unproblematic" (p. 27). Further, she questions assessment based on students' writing because of their lack of facility with mathematical forms (p. 35).

Morgan (1998) is interested in helping students improve their writing, suggesting that when it appears that students are improving "naturally", they are more likely learning "the features of the genre that will be valued by their teachers" (p. 42). She counsels teachers to remember that even when students are asked to write for an imaginary audience, they know that their audience is ultimately teachers. A teacher's assessment of their work naturally directs students to reflect the teacher's values in their writing.

Teachers tend to value more easily assessed approaches, which can conflict with student creativity (Morgan, 1998, pp. 117-119). She finds such features sought after by the teachers in their assessment include tables as evidence of organization (p. 152), algebraic generalization (p. 162), thinking verbs (e.g. *predicted*) rather than sensing verbs (e.g. *saw*) (p. 165), explanation but not too much of it (p. 170), and conventional use of the mathematics register (pp. 171-173). Teachers do not handle the assessment of "different" or "creative" student texts uniformly. Teachers seem to compose explanatory narratives in their interpretations of any student writing (p. 182).

Morgan (1998) ends by suggesting that teachers be explicit with their students about language, genres of text, and forms that are valued. With this open dialogue, she hopes that together they can develop critical language awareness (p. 209).

Blowing Leaves – Investigation Outside of the United Kingdom

The final portion of my review of literature surrounding the "investigation" is set outside the context of the United Kingdom. Imagining again the tree in the field, I see leaves blowing around throughout the field. It becomes extremely difficult to locate from which tree a particular leaf comes. Many of the trees in the field are of the same kind.

Indeed, many of the trees come from the same ancestor. Here I describe some literature from the field of mathematics education that reminds me of the British "investigation" tree.

The word *investigation* appears in current North American high school mathematics textbooks, but in most cases in which I have seen these investigations they seem to direct students toward a particular result to be discovered. Because they are directed, I do not associate them with the explicitly open-ended investigations seen in the United Kingdom. Although the word *investigation* is used differently in North America, there is a literature that values investigative settings – settings in which students are encouraged to engage in dialogue surrounding open-ended problems rich in complexity.

Considering the Classroom Environment

Schoenfeld (1988) believes that school mathematics is "most appropriately viewed as simultaneously comprising both cultural and cognitive phenomena" (p. 82) and describes one teacher who appears to be sensitive to these two phenomena in his pursuit of supporting student sense-making. He asks, "How can we create classroom environments which are microcosms of the right mathematical culture?" (p. 87) and seems confident that anyone who would take up this research challenge would find it a rewarding experience.

Borasi (1992) embraces Schoenfeld's challenge. She conducted a ten-lesson mini-course on mathematical definitions for two students who had missed a large portion of their regular mathematics class. In their interactions "both students took advantage of almost every opportunity for creativity offered to them" (p. 148) and developed an appreciation for this discipline which they had formerly despised. Borasi suggests that the goals of mathematics teachers ought to include an aim to help students embrace the humanistic aspects of the discipline. To do this, the teacher ought to promote student interaction and present "real" problems – ill-defined problems that admit ambiguity and genuinely different solutions. She carries the hope that such an approach can lead students to become reflective and critical thinkers in the field of mathematics and to think of themselves as real mathematicians.

Lappan (1997), a former president of the National Council of Teachers of Mathematics (NCTM), interprets the intent and substance of the NCTM's (1989) Curriculum and Evaluation Standards for Mathematics to include investigations, which she describes in this way:

> Deep understanding is best promoted by posing problems and questions, and then skilfully guiding problem solving and discourse so that students' ideas are constantly probed and pushed toward more powerful mathematical realizations. (p. 210)

It seems from this description that she has in mind an entire class thinking mathematically together. In the United Kingdom, students tend to work in small groups with limited teacher guidance in their approach to the given tasks. By contrast, there are plenty of examples of North American research describing classrooms in which students make and justify conjectures and question each other under the teacher's orchestration.

Forman, McCormick and Donato (1998), for example, analyze the discourse between a teacher and three of her middle-school students. Their conclusions suggest that when students are engaged with an investigation, the teacher ought not to be the regulator of discussion. Instead, the teacher can encourage multiple solutions and "help students learn how to assist each other's learning" (p. 317) by moving beyond initiation of discourse and by becoming an orchestrator of discourse. Perhaps an interest in teacher orchestration moves discourse to the inclusion of the entire class because of the impossibility of a teacher orchestrating each of many groups' dialogues.

Lampert (1990) describes her teaching experience in a grade five class using a starting point activity that could easily be classified as an "investigation" in the United Kingdom. Here she describes her orchestration of student dialogue:

> I gave [my students] problems to do, but I did not explain how to get the answers, and the questions I expected them to answer went beyond simply determining whether they could get the solutions. I also expected them to answer questions about mathematical assumptions and the legitimacy of their strategies. (p. 38)

The behaviour Lampert elicits from her students seems to be the same kind of activity hoped for in investigations done by students in groups with minimal teacher intervention.

Chazan and Ball (1999), with their study of two classroom environments, provide insight for teachers encountering disagreement and argument between students actively involved in mathematical discourse. They move beyond the negative exhortation not to tell. They recommend that the teacher introduce "mathematics which up until [the point of argument] was not part of the conversation under consideration" (p. 9). The argument in Chazan's class involved older students' disagreement unaccompanied by reflection. The argument in Ball's class, also attributed to a lack of reflection, is between the "students and the mathematical community, represented by the teacher" (p. 8).

Considering the Nature of Mathematical Problems

Although the literature cited so far focuses on the nature of the classroom environment in which investigative discourse takes place, another source of concern is the nature of the problems used to engage this discourse. Kilpatrick (1987) reminds us "that a problem is not a problem for you until you accept it and interpret it as your own. One person cannot give a problem to another person" (p. 124). He focuses attention on the students' experience of problems posed by teachers, suggesting that students may perceive problems as rather contrived. Noss (1983), out of experience with Logo investigations in British mathematics classes, shares this concern: "The essential power of discovery lies in the sense that the new idea belongs to the learner" (p. 9).

Nemirovsky (1996) agrees in his criticism of "real-world" problems. He explains that the "real contexts are to be found in the experience of the problem solvers" (p. 313). He asks, "If we think that algebra relates to factual issues of political or personal relevance, why invent problems?" (p. 300). His concern centres on his understanding that what seems real to a teacher or textbook author may not be real and compelling for students. He supports "real-life" problem posers in their interest in complex contexts, but argues, "Complexity, rather than being an exogenous factor defined by the problem, emerges from the qualities that surround the students' experience of the problem" (p. 312). He argues that concern for decontextualized problems is unfounded.

Gerofsky (1996) also argues against the apparent need to contextualize problems. She finds that teachers expect their students to ignore the "alibi" – the story structure on which the problem hangs. She describes the typical expectation for student work on a contextualized problem:

> Too much attention to story will distract students from the translation task at hand, leading them to consider "extraneous" factors from the story rather than concentrating on extracting variables and operations from the more mathematically-salient components. (p. 38)

She asks if this hypothetical "translation" or "transformation" exercise should be considered an important part of mathematical thinking and worries about the effects of training students to ignore the context of the problems they will encounter in their world.

Wilensky (1991) proposes a creative alternative to the typical understanding of concrete contexts for problems. In consideration of Piaget's description of developmental stages, from concrete operations to the abstract, Wilensky feels compelled to reconsider the idea of the "concrete". The normal sense of the word is a relic of positivist thinking and does not fit the *constructionist* paradigm that he supports. "It is futile to search for concreteness in the object – we must look at a person's construction of the object, at the relationship between the person and the object" (p. 198).

Wilensky (1991) suggests that we redefine the *concrete* as something that the individual connects to other objects. "This view will lead us to allow objects not mediated by the senses, objects which are usually considered abstract – such as mathematical objects – to be concrete" (p. 198). He then uses his new definition to consider how operations on fractions might be looked at "concretely" instead of abstractly.

In this vein, the type of problem posing suggested by Brown and Walter (1990) might be regarded as concrete and thus appropriate for starting points in student construction of understanding. They hope to "encourage a shift of control from 'others' to oneself in the posing of problems" (p. 1) – from the teacher as initiator to the student as initiator, in the setting of a mathematics classroom. With their dynamic fusion of mathematical activities and reflections, Brown and Walter seem to suggest that teachers ought to initiate mathematical activity with questions or statements that draw more questions out of their students. They propose that such an approach can address mathematics anxiety and promote cooperative mathematical thinking.

A Settling Seed

The two classes that I observed conducting "investigations" took place in Alberta. The literature from the United Kingdom describes similar student tasks to those undertaken by the students in my study set in Canada. This literature, then, provides examples of student discourse and insight into the tensions faced by the students and their presiding teachers in an investigation setting.

The North American mathematics education culture, which is partly influenced by its literature, has more directly informed the present mathematics classroom culture experienced by the students in my study. The background experiences of the teachers in my study will fit more closely with the discourse in this part of the literature.

The British mathematics education discourse surrounding investigations, together with the culture of North American mathematics pedagogy, inform a careful analysis of the data from my study. I witness a slightly foreign seed drifting into new territory from the tree in the field – the British investigation experience.

Chapter 3 – Planting a Seed of Hope: Method

For this research, I studied two classes of grade ten students that engaged in mathematical exploration both prompted by the same two investigations that I supplied. I see comparisons between my research experience and the experience of my research participants. Both my investigative research activity and my construction of the investigation prompts addressed by these students are hope-based. They are based on a belief that immersed exploration in open and complex territory yields harvests of understanding and awareness.

As I developed a plan for addressing my research interests, I became aware of my inclination toward hope-based learning. My recollection of an image from my family's time in Swaziland demonstrates to me that I have been reflecting on the value of hope for some time.

Close to our adopted family's homestead in Swaziland there was a rusted car body lying beside a road. Tiny pink flowers grew through its grille. To focus on a single one of these flowers was to see insignificance embodied. It was only one of many equals around it. Its fragile body grew amidst junk beside an unnamed road in a small town settled by displaced people. It was in one of the smallest countries in the least respected continent in the world. Its home was in the dirt and it could not move.

The car, on the other hand, embodied significance. It was produced in the most important car-manufacturing city of the world's most powerful nation. It was a symbol of power, wealth, strength and freedom. This automobile was imported to Africa for a person with riches and status. The engineering marvel had been pampered, washed, greased and polished. It would drive everywhere, drawing admiration from everyone.

But the flower lived and the car was dead. Yes, the car once roared with life and dominated its environment but now it was dead and a pollutant. The flower would also die, but it would continue to live. In its death, it would plant seeds for the future, for more flowers, beautiful like itself.

I often recall this image when I watch rush-hour traffic – a lonely world that is dense with human population, with every person insulated from the others by the walls of their machines of independence. In reflection, I compare this kind of transportation with what a Swazi friend called "the people's transit" – walking. Some of my favourite Zulu songs employed the metaphor of walking to describe the way we can live in awareness of our connection with others.

As I find myself walking around in the places I live and work, I am often aware of the humility of this and other Swazi friends. While they were generous in small ways to small, typically undervalued people, the impact they made on their environments was enormous. I think of their humble approach to life as I think of flowers – they release small seeds knowing that these seeds carry with them tremendous potential for vibrant growth.

My research planning has been significantly influenced by my valuing of small things and the hope that accompanies these values. I focused on two small places – two classes that might seem insignificant relative to the extent of Alberta's education system. I chose small experiences with the confidence that they would generate ideas and connections to other research. For me to choose such experiences was to embrace my subjectivity, my interconnectedness with the people I would observe. While I planned to involve myself with my participating students and teachers, I would try not to interfere with their choices. With the choices I made in my research planning, I chose to plant a small seed, to take care of it and to watch how it would grow.

For the fruition of their plans, gardeners are dependent on many factors beyond their control. A garden's growth is dependent on sufficient moisture, sufficient drainage, sufficient sun, sufficient soil nutrients, sufficient time for maturity, and protection from parasites. Some of these factors can be manipulated to a certain extent, but not completely. The gardener can only plant untested seeds, and hope they are viable. The gardener's hope lies in experiences of previous harvests.

Like a gardener's planting, this research is rooted in hope and experience. Many of the factors influencing the events that I planned to observe would be beyond my control, because the events would be part of a complex system of classroom, school and scholarly expectations. But my teaching experiences, and my vicarious experiences gained by reading other hope-grounded research, allowed me to enter into this endeavour with an expectation of something wonderful.

Preparing the Ground – Selection of Mathematics Classes

The first two teachers I contacted about participating in my study welcomed my involvement with their Pure Mathematics 10 classes. Each of these teachers was a prior acquaintance. Mr. Penner taught in a large urban high school with a grades 10 to 12 student population of about 2,000. Mrs. Foster taught in a Kindergarten to grade 12 school of about 600 students located in a town that serves as a rural centre and also houses families with members who commute daily to a large urban centre. Both Penner and Foster seemed to desire experience with projects in their Pure Mathematics classes.

I contacted these two teachers in January 2001 so that they would have time to include the projects in their second semester planning. My participation in the rural school extended from the beginning of April until the beginning of May. In the urban school, I participated from mid-April to mid-May.

In both settings I observed a few classes before students embarked on the investigations I provided. I also observed a few classes between the days involving these projects. These times afforded me the opportunity to compare dialogue from investigative work with more established dialogue patterns and also allowed participants to become more familiar with my participation and the presence of audio and video recording equipment.

Both classes followed the same basic timeline with regard to the projects: Figure 1 outlines this progression. A few schedule changes were forced by unforeseeable time conflicts. My second interview with Mr. Penner was delayed by one day, and my second set of interviews with the rural class was delayed almost one week.

Seeds of Growth – The Investigative Projects

Figure 1 indicates that both classes explored “Playing with Squares” before “Parallel Division”. Mr. Penner’s class used a slightly different version of “Parallel Division” from Mrs. Foster’s class, because he inadvertently photocopied a version I had given him months earlier. In this earlier file, he also found two scoring rubrics, the one he used for “Playing with Squares” and an exact copy of it minus the creativity criterion. He told me that he chose the one that did not evaluate creativity in order “to try it out”.

School Days	**Rural Class**	**Urban Class**
1	Observe class.	
2	Observe class. Mrs. Foster distributes and discusses the scoring rubric with students.	
3	Observe class.	
4	**"Playing with Squares"** project.	
5	Observe class. Mrs. Foster asks students to write whatever they want to tell her about the project.	
6		
7		
8		
9	Observe class, and interview Mrs. Foster and selected student groups.	
10		
11		
12		Observe class.
13		Observe class.
14	**"Parallel Division"** project	Observe class.
15	Observe class. Mrs. Foster has students anonymously complete a questionnaire about their project work.	1st period – Observe class. 2nd period – **"Playing with Squares"** project
16		Observe class.
17		
18		
19		
20		1st period – Observe class. 2nd period – Interview Mr. Penner and selected student groups.
21		
22	Observe class and interview Mrs. Foster and selected student groups.	
23		
24		
25		1st period – Observe class. 2nd period – **"Parallel Division"** project
26		
27		
28		Observe class.
29		
30		1st period – Observe class. 2nd period – Interview Mr. Penner and selected student groups.

Figure 1. Timeline for class participation

Figure 2 presents the projects, including the two slightly different versions of "Parallel Division". The scoring rubric is given in Appendix 1.

When constructing these projects, I had a few simple goals in mind. I believe that these criteria can form fertile ground for the construction of mathematically rich student tasks:

- simple instructions;
- open-ended questions;
- not embedded in a real or hypothetical story;
- a promising connection to the school course of studies, the Pure Mathematics 10 standards from Alberta Learning (2000) in this case.

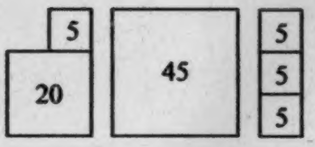
My earlier experiences developing projects for my own students and for sharing with other teachers has demonstrated for me the value of short and simple instructions. I prefer that students are not distracted from their mathematical thinking by long and complex instructions.

Open-ended instructions are the basis of the British investigations described in Chapter 2. They set students on a creative path of exploration, freeing them from the restrictive sense that they need to produce an expected result.

Ironically, Nemirovsky's (1996) critique of real-world problems helps me understand the potential value of context-based problems. However, with Gerofsky (1999), I am disturbed by "throw-away" contexts. I think that, when alibi stories are part of mathematical problems, students ought to be encouraged to consider the contextual implications in conjunction with the mathematics. If they are not encouraged in this way, I would hope that they would discuss the reasons for ignoring contexts. I feel that,

Playing with Squares

The 45 cm^2 square is the exact same height as the two stacks of squares beside it. The squares in the stack on the left have areas of 5 cm^2 and 20 cm^2. Each of the three squares in the stack on the right has an area of 5 cm^2. For this assignment, the area of any square should be a natural number when measuring in square centimetres.



Stacking Squares

- Find stacks of squares that would be the exact same height as a square with area 72 cm^2.
- Are there any squares which could have no stacks that are the exact same height? Explain.
- Explain how to find the stacks that would match a given square in height.

Add a dimension

- How would all this work for cubes instead of squares?

Present your findings on an 11 x 17 inch sheet of paper.

Parallel Division

The coefficients in this polynomial division correspond to the digits in the numeric division.

Compare the division of $(2n^3+7n^2+8n+3)\div(2n+3)$ with $2783\div 23$.

- What do you notice?
- Can your observations be generalized for all polynomial divisions? Explain.
- What are the results of replacing numbers or signs in the polynomial division?

Present your findings on one 11 x 17 inch sheet of paper.

Mrs. Foster's version.

Parallel Division

The coefficients in this polynomial division correspond to the digits in the numeric division.

Compare the division of $(2n^3+7n^2+8n+3)\div(2n+3)$ with $2783\div 23$.

- What do you notice?
- Can your observations be generalized for all polynomial divisions? Explain.
- Comment on some interesting values of *n* in the given polynomial division.

Present your findings on an 11 x 17 inch sheet of paper.

Mr. Penner's version.

Figure 2. The investigative projects

without an awareness of the artificiality of stories with throw-away actors and events, their regular use devalues persons and characterizes mathematical reasoning as unconcerned with human diversity. Pure mathematics investigations can draw attention to mathematical reasoning without devaluing human experience.

On the other hand, I also resonate with Confrey's (1995) warnings about abstraction. She notes two historical roots of the word *abstraction*: its connection to "political oppression and elitism", and also its connection with the Catholic language of absolution from carnal imperfections, with this tradition's implied privileging of the mind over physical experience (p. 40).

Confrey (1995), however, describes mathematical abstraction that does not assume disembodiment. She supports mathematical experiences that integrate practical activity with sign use, that value multiple forms of representation and that involve participants in the action of abstraction (p. 40). I suggest that "Playing with Squares" is a strong example of her three descriptions, especially the first two, because it asks students to relate physical representations to the abstract signs that are more typically favoured in mathematics classrooms. It asks students to let their more concrete diagrams inform their sign use and vice versa. The "Parallel Division" task does not fit her descriptions as well, but I suggest that this investigation does draw students' attention to their conjectures. If students are attending to their *action in the abstract environment,* they approach Confrey's call for attention to the *act of abstraction itself.*

Both of these projects are derived from curricular expectations of the participant students. My construction of "Playing with Squares" is based on my visual representation of radical addition. Because of my lack of experience with making visual representations of mathematical relationships, it took me considerable time and effort to find such a representation for the addition of radicals – a procedure that seemed to be divorced from physical connectedness. The students' instructions aim to direct student attention to the wealth of possibilities available to them for representing mathematical ideas.

"Parallel Division" arises out of my many attempts to teach polynomial division. I have always compared it with whole-number division and I have always been careful to select examples that make the parallels obvious. As I developed these projects, I wondered why I would be so careful. Did I really want to shield my students from unnecessary complexity? This investigation is constructed to draw students into complexity. Students are expected to make and test conjectures.

I invite readers of this thesis to try these two investigations before reading my presentation of the data. I believe it is easier after exploring the territory laid out by these projects to understand the feelings and actions of my participant students and teachers.

Watching for Growth – Methods of Observation

To record what happened in these classes, I took handwritten notes and made audio and video recordings. To include the perspectives of my participants in my interpretations, I conducted formal interviews with them. I also interviewed many of them informally with short conversations during their work. In our formal interviews, students and teachers reconstructed their experiences, and encountered the inaccessibility of the past. My memory of what happened in these classes is not only another record of the events, but also influential in my interpretation of the audio, video and textual records. I believe that, in any act of interpretation, the primacy of memory is unavoidable.

For each class period I observed, I had the teachers wear lapel microphones and I had a video recorder in a corner of the room. For classes in which projects were not being done, I included an omni-directional microphone in the middle of the students' area. For the project work, I included a tape recorder in the centre of each group's gathering. I wore a lapel microphone to record my conversations with individuals, though I often forgot to turn it on. A number of my important exchanges are only recorded in my paper notes.

I tried to avoid participation in the classroom dialogue. However, during the project work I found it difficult to construct conversational questions that did not direct students' mathematical thinking. Ainley (1999) suggests that a classroom researcher needs to choose between acting passively and acting as a catalyst. A passive role is more conducive to observing events as they would presumably unfold in the researcher's absence. Although I remained aware of Ainley's advice, I found that, with passive observation, I was often not able to observe that which interested me most – students' thoughts and feelings. If students were not sharing their feelings without prompts, I felt drawn to catalyze their conversation.

I wonder now on what basis I thought that my chosen groups would provide the most interesting material. I think that even at the times of my choosing I was unaware of my criteria. The best explanation I can provide is that something in each of these groups' work captivated me. I see now that, in most cases, I had already been captivated while the groups were working. Somehow I was drawn more to some groups than to others. Perhaps something I did not expect occurred in these groups. Perhaps they exuded more energy. Perhaps they were more willing to engage in conversation with me as I passed by. Perhaps I was drawn to the groups that the teachers approached differently from other groups.

The direction of my attention in research reminds me of my apparently inexplicable choices of direction when I engage in mathematical exploration myself. After being engaged, I am able to spin an explanatory narrative, but during the experience my attention is not directed at meta-analysis. Rather, particular questions and problems captivate me, causing me to ignore other possibilities that have the potential to be equally interesting.

Every time I listen to the audiotapes of the classroom interactions, new aspects captivate me, but the events that consume me – the situations that I find most interesting even now – are still the ones that captivated me when I was experiencing them in person. I expect that these events will remain closest to my consciousness. My semi-structured interviews with student groups and teachers contain a similar mystery. Why did I divert from my plans laid out in my interview guides at these particular times? What kinds of exchanges captivated me enough to ignore my plans? Although I prepared interview guides for both teacher and student interviews, I often strayed from the guides to pursue interesting strands introduced by interview participants.

My interview guides were significantly influenced by Ginsburg (1981) and Gattegno (1981), both of whom display substantial confidence in students' awareness and ability. Ginsburg, in particular, applies this confidence to the art of interviewing: "Put simply, if you want to know what someone is thinking, ask him" (p. 7). I did not try to trick the students and teachers I interviewed into revealing something that might be otherwise inaccessible. Instead, I freely told them about my interests and asked them to tell me what they noticed about their investigative project work, taking into account my interests. To elicit my participants' interpretations of the events of interest, I played audio recordings of group work in some interviews. Besides providing productive prompts for student explanation, this strategy proved entertaining for students.

In addition to these audio and visual records, I also collected text. I collected copies of the posters made by students during their investigative projects and the teachers' evaluative notes on them. The day after the first project, Mrs. Foster asked her students to write for five minutes whatever they wanted to write about the previous day's experience. After the second project, she distributed a questionnaire to draw feedback from her students regarding issues that she felt were important. She provided me with copies of both sets of this data. During the first project, Mr. Penner noted that students were not writing on their posters as much as he would have liked, so he asked them to submit their rough work along with their posters. He provided me with copies of this work as well. Mr. Penner's realization that the posters were inadequate representations of his students' thinking is reminiscent of the motivation behind Morgan's (1998) study.

Gentle Gardening – Ethical Considerations

Before my classroom involvement, participating students and teachers provided me with signed consent forms. I provided a brief description of my intentions for the students' parents and teachers as well as an invitation to direct questions or concerns to me. The letters and consent forms are given in Appendix 2. Students and teachers were invited to opt out any time they were uncomfortable with participation. During the investigative work itself, the participants in one group exercised this option by turning off their tape recorder periodically.

All the names of participant teachers and students in this research are pseudonyms. For female participants, I use typical female names and for males I use typical male names. Although I chose pseudonyms with ethnic parallels to the participant's real names, I recognize that many students' real names do not mesh with their apparent ethnicity. A Korean student who uses a European name, for example, would be assigned a European pseudonym according to my scheme.

Harvest

My gardening experience has taught me that I can never be sure what to expect from a seed. It was similar with this research. Although I had expectations of what might come of my investigations, I continually reminded myself to wait and see.

This waiting reminds me of Ping, a young boy in Demi's (1990) story The Empty Pot. In this tale, the aging emperor of China gives a flower seed to each child in the land, saying, "Whoever can show me their best in a year's time will succeed me to the throne" (p. 7). In a year, each child came with a splendid flower, but Ping comes with an empty pot. When the emperor asks him about his empty pot, Ping replies:

> I planted the seed you gave me and I watered it every day, but it didn't sprout. I put it in a better pot with better soil, but still it didn't sprout! I tended it all year long, but nothing grew. So today I had to bring an empty pot without a flower. It was the best I could do. (p. 28)

For his truthfulness, Ping is rewarded with the kingdom, because all of the emperor's seeds had been cooked and could not possibly have grown.

When I think about empty pots, I realize that they are not empty at all. If I were to pour soil out of a pot, for example, would it be empty? No. It would be full of air. We might call it empty, because it does not have the thing we want in it or the thing we expect to be there, but really when we pour soil out we are simply exchanging soil with air. Ping did not present an empty pot to the emperor. His pot held truth.

Before observing the two participating classes, I had tried to prepare myself for the possibility that I would find nothing, or, in other words, something different from what I expected. Before engaging with these people, I had thought of such a result as emptiness even though I was aware that beautiful surprises often await people who hope. Once I began participating in the life of these mathematical environments, it no longer mattered what I was expecting. What mattered was what I was experiencing.

I stuck to my planned course of action, digging with the hope of revealing that which I expected, even though I was not finding what I expected. The apparent emptiness, or insignificance, of the answers to some of my research questions no longer bothered me. For instance, the most fascinating parts of my interviews were the places where we strayed from my planned questions.

The final stage of my research action was my interpretation of the classroom experiences. At the interpretation stage, I synthesized my classroom observations with my experiences outside of these classrooms. I found an unexpected fullness after planting a seed in preparation for a particular result. Although my expectations did not come to fruition, I see the result of my investigation as fullness. In the next chapter, I describe the way of seeing that has for me become a most interesting and unexpected fruit of the seed I planted with my research plan and activity.

Chapter 4 – Ways of Seeing a Mathematical Place

During my engagement with the two participating classes, I focused my attention on attempting to notice cultural shifts and the manner in which students and teachers responded to these shifts. This relatively narrow perspective was constrained even more by the highly finite nature of my existence in these spaces. I could only watch one group or one person at a time. Since then, I have had plenty of time to think about what I observed and I additionally have access to audio, video and textual records.

Now, with time to consider and reconsider the events in the two classes, I find that I have been drawn into a sense of sympathy with the participants. I no longer feel like an objective outsider trying to measure change or effect. Instead, I feel pangs of familiarity with their disorientation in a mathematical setting that was new to them. This current sense of commiseration is the one that dominates my way of interpreting the data – my way of seeing. Thus I describe this way of seeing before reporting on what I saw happen in the classrooms.

As the students and teachers involved themselves in a mathematical experience that was somewhat foreign to them, I watched them wonder what was expected of them and struggle to find words for their mathematical ideas and questions. I was reminded of my feelings of disorientation in the early stages of my three-year tenure in Swaziland, when I could not understand people's expectations of me and could not find words for my questions about the things people around me took for granted. This excerpt from a story I wrote about an experience more than a year into my time in Swaziland describes my disorientation:

> It was a pleasant morning and I was feeling good – I was beginning to feel that I belonged among my Swazi colleagues, students and neighbours. Walking to school I was greeted by passers-by, *Yebo thishela* (Hello, teacher), and I responded appropriately in SiSwati. I was becoming accustomed to things that had seemed strange to me, like the school's morning assembly in which students sing a Christian chorus and recite the Lord's prayer. I was already aware of some of the finer points in Swazi etiquette, so that when a colleague gave me my mail this day I received it with two hands and a slight bow of the head. I felt like I was beginning to understand not only the school culture, but the wider Swazi culture as well.
>
> On this day I confidently told my Form 4 (Grade 11) students that they would write a test the next day. It wasn't until the end of the day that I realized my mistake. A student asked me, "How can we write the test tomorrow when the school will be closed due to the *sibhaca* [traditional dance] competition?" I suddenly felt like a foreigner again. I was a little annoyed that my students hadn't told me about my mistake at the time, but I also knew that in Swazi culture it is extremely rude to contradict an elder, especially a teacher. I was a little annoyed that my colleagues hadn't informed me of the school's closure, but they thought I knew.
>
> I had known about the competition but had assumed my class would be exempt because it had no dancers. I knew that community is much more important than individuals in this culture, but only after this incident did I understand how this belief applies to events. When individuals represent the school community, everyone supports the events and attends if possible. And by no means should we do anything important when these people are absent on our behalf! (Wagner, 1998, p. 5)

My feelings of disorientation had been even stronger in the first months and were worse yet in the first weeks in Swaziland. I cannot even describe my disorientation in the earlier stages of my family's cultural immersion. I do, however, remember feeling like my new neighbours did not understand the intentions and feelings behind my actions. Certainly, I did not understand their motives and actions. We were strangers.

My own dominant experience of unfamiliarity is this experience in Swaziland, so I draw upon it in my interpretation of the events in the classes I observed. With this choice in mind, I recognize that many people have probably experienced the disorientation that accompanies immersion in foreign places. I felt such disorientation when I began undergraduate studies, when I began my first year teaching, when I was an experienced teacher beginning in a new school, when I joined new soccer teams and when entering graduate studies. I have even felt it when discussing with close friends topics we have not discussed before. My first experience of disorientation with a foreign place immediately followed my birth. Although I do not recall my experiences as an infant trying to make sense of my place in the bright, cold world of hunger, thirst and relationships, I have watched my own children suffer and grow through this kind of disorientation.

Ways of Seeing

At the 2001 Canadian Mathematics Educators Study Group, I participated in a working group entitled "Where is the mathematics?" (Mason and Muller, 2001). During this time of mathematical play and reflection, I noticed that it is easy to talk about a place's whereabouts when I am not there, but altogether different when I am actually present in the place. Where I position myself influences the complexity of questions of location. I can see the boundaries of places from which I am distant, but boundaries look different for the places in which I dwell.

When I lived in Swaziland, for example, if someone asked me where Canada was I could say it is immediately north of the United States, the northern half of North America. Easy. Sitting here in Edmonton the question is quite different. Where is Canada? It is right here. It is everywhere I see and everywhere I go – in all directions. Indeed, because I am a part of Canada, it is both inside me and all around me.

It is the same with mathematics. If I look at a mathematical text, I can identify the mathematical aspects present there. I might say, "Look at this part of the explanation; here is the mathematical essence." But when I try to locate the mathematics in my own life, it permeates every place I move and see, and it is a part of who I am. Brown, Hewitt and Mason (1994) recognize the benefits of different ways of seeing mathematics. They use the metaphor of filters to describe three different ways of seeing. In photography, if I simply change filters, I do indeed see different pictures. However, all these pictures are still taken from the same perspective. In this discussion concerning ways of seeing I focus on positioning in the mathematical place. Change of position implies change of perspective.

A Mathematical Place

I think about a mathematical place as a place of discourse. The Greek word *topos* carries a broader meaning than the English word *place* (OED, 1989, "topo-"). *Topos* can include both a physical location and a topic of discourse. It is a root of both *topology* and *topic*. Even in English, it is not uncommon to use the language of place to describe relationships. When a hockey player is adjusting to new teammates we might say he is finding his "place". People take on "positions" of authority. Similarly, the language of place is used to describe discourse or conversation. When I am telling a story and I lose my "place", I might say "Where am I going with this?"

When I talk about a mathematical place, I think of its physical place, the topic of discourse, the participants' place in relationship to others in the discourse, and even its place in time. Each of these kinds of places is part of the mathematical *topos*. In Figure 3, I draw a geometrical picture that describes in part the multiple dimensions involved in my understanding of place. Time does not appear in this picture because I am unsure how to model four dimensions.

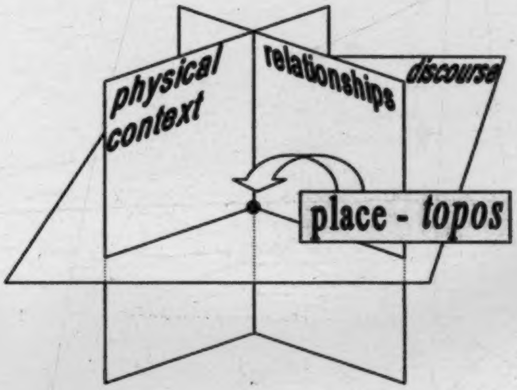


Figure 3. The multidimensional place

Words that refer to physical spaces represent not uncommon metaphors in text referring to mathematical discourse. We have "fields" and "areas" of study, and "landscapes" of investigation. We "move" from one topic to another when we "follow" curriculum. We "explore territory".

One problem with using an expanded notion of "place" as I do is that the physical and temporal place pre-exists the topic of discourse and the people engaged in the discourse. Some might argue that even the topic exists before people are engaged in it. My construal of the temporal–physical–relational–topical place comes close to Piaget's proto-space and proto-time (see von Glasersfeld, 1995). A physical and temporal place, or even a topic of discourse, is meaningless and unimportant to me until I experience it; it does not exist for me. Once it becomes part of my experience then it is only significant to me in the way that I experience it, that is in connection with the human relationships that accompany it.

Being in a Foreign Place

With the two classes involved in my investigative projects, I found evidence that the students and teachers were in a place new to them, a foreign place. They experienced something new in each of the four ways that I am considering the mathematical *topos*.

They experienced a new discourse. True, the mathematical topics were not unfamiliar, but within these topics they were confronted with tasks that were not familiar to them. They were contending with open-ended questions that opened up a space of possibility in which they felt unsure where to step. Teachers and students alike wondered what to do. They wondered what to say and in what pedagogical and mathematical direction to go.

With this change in discourse, the classroom actors did not know how to relate to each other. They had little or no experience operating in an open investigative mathematical landscape. They shifted around uncomfortably trying out different ways of relating to each other. Even the time and physical realities seemed different. Here they spent an hour on one question, when they were more accustomed to valuing speed and efficiency.

Instead of being stacked neatly in rows with wide spaces between them, students were bunched together, leaning toward each other, facing each other. Their teachers found the need to develop new patterns of moving about in this new layout. Their usual vertical and horizontal beats were replaced by frequent stops and wending paths.

Teachers and students alike suffered disorientation because of this foreign place. Although they were the same people, using the same furniture, in the same timeslot and considering the same general areas of mathematical study, everything was different. It did not feel like mathematics. They had entered a new territory. The territory was fertile, calling out for viable seeds, but the new inhabitants were inexperienced gardeners. Their previous mathematical experience was precooked. Now they struggled to grow their own fruit. They were accustomed to buying packaged and prepared food at supermarkets, forgetting about seeds, farmers, fertilization, weeding, unpredictable weather and harvest.

Ways of Being

I have traveled extensively, and I have been immersed in a foreign community with little opportunity for communication with other expatriates. Because of these experiences, I am interested in different ways people position themselves in places new to them. At various times and in various places I have, to some extent, been a tourist, a colonialist, a missionary, an aid worker, an advisor, an ambassador and a participant. The two ways of being that I find most interesting with regard to mathematical places are tourism and immersed participation.

When I was living on the black side of a racially divided mining town in Swaziland, I sometimes watched tourists passing through this country that was home for my family and myself for three years. I watched these tourists rush along the highway in air-conditioned buses, briefly stopping at "must-see" sites, sometimes stopping at the roadside to hear a guide's description of a "typical" nearby homestead.

As I watched these people experience this place in this particular way, I imagined what they might say to their friends at home. I have often listened to people describe with apparent authority places they have only briefly visited – places they have experienced like these tourists in their insulated coaches. I also wondered what the tour guides might be saying to their charges in the buses. These guides are not likely to have experienced in any depth the places they described authoritatively. Even if a guide had in-depth knowledge of a place, there is no way he or she could possibly communicate it to the tourists within their constricted itinerary. I had difficulty making any definitive statement at all about this place in which I was living, especially after more than two years' continuous experience. For me, a participant in the life of the place, it was complex. For the itinerant tourists and their guides, it was simple.

I suggest that mathematics teachers too often act as tour guides, hurrying students from one "must-see" curriculum outcome to another, uninterested in the connecting places between these sites and unconcerned with the greater whole to which they all belong. Because such tour-guide teachers are not likely to have recently been captivated by a problem in a mathematical place, they may be unaware of the shallow experience of mathematics they provide for students. Or, like the tour guide who stops the tour bus beside a disadvantaged Swazi homestead without getting out, perhaps teachers fear commitment to the experience of mathematics. They worry about the time that is likely to be consumed and the subsequent disruptions of their itinerary and their understanding of their position in the mathematical world.

Problems – What is Real in Mathematics?

Thinking about mathematics as a place in which one could be a tourist or a participant has also prompted me to reflect on the nature and experience of problems. In Swaziland, a guide might tell tourists that there is a problem with AIDS, for example. An outsider might easily provide a solution: "Condoms are the obvious answer." But for the people in the community there are no easy solutions, only complicating factors. One could question even the solutions that seem to have some validity – solutions that are in fact helpful because they address parts of the problem. The participant knows that the problem is ill-defined, full of ambiguity and complexity.

For example, whose authority could convince sexually active Swazis to use condoms? Expatriates? No. Africans are realizing more and more how many of their problems have their source in foreign "benevolence". Perhaps local religious or tribal leaders might be able to convince people to use condoms. But who will convince these leaders to do this? Expatriates? The complications I introduce with these questions are not the only complications.

For the participant, a problem is in fact a problem. Participants are captivated and consumed by the problems of their place because they belong to the place as much as the problems do. What makes a problem "real" for outsiders is their perceived ability to "solve" it. The problem gives them an opportunity to demonstrate their knowledge and "objective" judgement. By contrast, the participants' inextricable connection to the problem is what makes it real for them.

My hope is that all students experience in their mathematics classrooms a place in which they can become captivated by the mathematics as participants. Borasi (1992) shares this goal, suggesting "that mathematics classrooms become 'communities of learners and thinkers' who are working together" (p. 170). In such communities, "mathematical problems are ill-defined" (p. 168) and thus reflect "the complexity of real-life problematic situations" (p. 191). They reveal the humanistic aspects of the mathematics discipline – uncertainty, ambiguity, personal judgment, cultural values, purpose and context.

I am not sure that I understand what Borasi (1992) means by a real-life problematic situation. In Nemirovsky's (1996) deconstruction of the notion of real-life problems, he suggests that different things become real to different people. A mathematical problem becomes real for students, he explains, when they find themselves wrapped up in its complexity. Even *complexity* is as evasive as *real*. It is defined by the qualities that surround the students' experience of a problem (p. 312). Here, Nemirovsky comes close to Wilensky's (1991) proposed redefinition of *concrete,* in which he suggests that something's concreteness is not inherent to it, but rather dependent on its level of connection to the experience of the observer. Both Nemirovky and Wilensky seem to suggest that a problem becomes real for a student when the student becomes engaged with it, consumed by it, immersed in it.

Nemirovsky (1996) and Wilensky (1991) both focus their discussion of reality and concrete experience on mathematical problems, but I believe that Borasi (1992) is thinking of the connection between mathematical problems and not-necessarily-mathematical problems. She attempts to unveil the nature of good mathematical problems by comparing them with "real" problems. This comparison is compatible with my way of seeing mathematical experience – as a complex place in which we can choose the depth of participation or the shallowness of tourism.

Ways of Guiding

Although I have already displayed my rather negative feelings about typical tour guides, I want to consider the nature of a good guide. I characterize the good guides that I have experienced in Swaziland as either outfitters or neighbours.

With confidence that I am adequately prepared to enter a new territory, an *outfitter* considers my readiness to be immersed and sends me off at the most appropriate time. Once I am participating in this new community, I look for people within the community to guide me in a different way. I look for *neighbours* with whom I can share insights and experiences of this place that is new to me.

When my family and I began our time in Swaziland, we were immediately sent to a rural district to live for two months with a Swazi family – a family we grew to love and depend on throughout our years in the country. They were our neighbours, even though we lived a two-hour drive away from them after these first months. The directors of the organization that sent us to Swaziland were our outfitters. They placed us into the familial relationship with this Swazi family and later into another Swazi community to live and work. Once we were engaged in our new dwelling places, these outfitters could do little more than remind us that they had confidence in us. Because they were not participating in the life of our new place, they were unable to give us helpful particular advice. For that we looked to our neighbours.

Both these kinds of guides have their place in non-hostile environments, but I would seek a different kind of guide in a place I experienced as hostile to me. If I needed to pass through a treacherous mountain pass and had inadequate knowledge and experience to feel sufficient confidence, I would want a guide who directed my every step – a *step-by-step* guide. I would hope that no teachers imagine the mathematical place in which they lead their students to be such a hostile place. I have had many students who seem to imagine that all mathematical places are hostile to them, and I wonder if this image of inhospitability is a result of learned dependence on *step-by-step* teachers.

If my guides were to constantly ignore or devalue my insight – the things I notice – then I might come to the conclusion that I am blind, dependent on my guides' ways of seeing and my guide's step-by-step advice. This dependence would be quite unlike the independence inspired by neighbour and outfitter guides who express confidence in my readiness to dwell and participate in the mathematical place.

A step-by-step guide becomes necessary when the important thing is to move successfully from one place to another in a place where successful mobility is hard to come by. For an environment in which I dwell continually, however, I do not desire dependence on step-by-step guides. Even if I recognize the interdependence of the people in my community, including myself, I feel that I need a certain degree of independence – enough to be able to make my own decisions.

Teachers who behave like outfitters or like neighbours seem to support students' immersion in mathematical places. By contrast, step-by-step teachers are like typical tour guides who insulate their followers from the complexities of the territories through which they move.

Modelling Different Ways of Being

Diagrams help me visualize structures of relationships. I use the following diagrams in my upcoming analysis of the data. Figure 4 describes a classroom in which the teacher is an outfitter. Such teachers send their students deep into mathematical territory to explore. Although these teachers are not involved in the specific exploration in which their students are engaged, they may have already explored the territory independently, or with friends and colleagues. The arrows point both ways between students indicating that they interact with each other in the mathematical place. The arrows linking students to their teacher are unidirectional, because students eventually report to the teacher. Even before reporting, student attention is partly taken up by the preparation for the reporting.

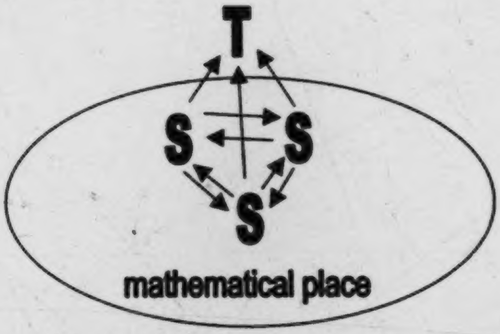


Figure 4. Outfitted immersion

Although the boundary around the mathematical place is represented by a line here, I visualize a fuzzy boundary. Where the mathematical place ends and where it begins are distinctions that make little sense to a participant immersed in a problem. I understand boundaries to be more important to people who desire a shallow or sheltered experience of a place. In my diagrams, I use a line to denote the boundary because no graphic depiction that I can think of does justice to the fuzziness of the boundaries as I imagine them. Marked boundaries make better sense in physical and temporal space than they do with the other aspects of my construal of place – topical and relational.

In this diagram, and in my subsequent diagrams that use the same imagery, the teacher is represented by the letter *T* and a student is represented by the letter *S*. The arrows represent the direction of attention. In Figure 4, for example, the unidirectional arrows between students and their teacher imply that the students pay attention to the expectations of their teacher who, in turn, is not paying attention to their experience of the mathematical place.

The kind of relationship to the mathematical place represented in Figure 4 seems to best describe the investigations popular in the United Kingdom. Jaworski's (1994) description of these investigations points to students acting independently of their teacher (p. 3). Along with this characterization, she acknowledges that teachers struggle with their involvement in such student-directed mathematics. Ben, a teacher whom Jaworski observes, describes his preferred role this way: "I like to be a manager of learning" (p. 144). His preferred interaction with students focuses on helping them to work together, not on approaches to the investigation. His intervention aims at affirming his students' readiness to be immersed in the mathematics without his support. He is amused by the irony of himself taking control in order to relinquish it.

Morgan (1998) notices that in mathematical relationships similar to the situation described here, students cannot turn their attention completely to the mathematical place. In my diagram, the arrows pointing to the teacher illustrate this distraction. Morgan demonstrates that students ultimately know that their teacher is their audience, and so they are necessarily distracted from complete immersion in the mathematical place.

The good guides whom I describe include neighbours as well as outfitters. Morgan's description of the inevitability of teachers interfering in students' completely immersed exploration suggests to me an alternative to the outfitted immersion model. If the teacher enters into the mathematical place with students and acts as a neighbour, the relationship would look quite different. Figure 5 illustrates this weave of relationships.

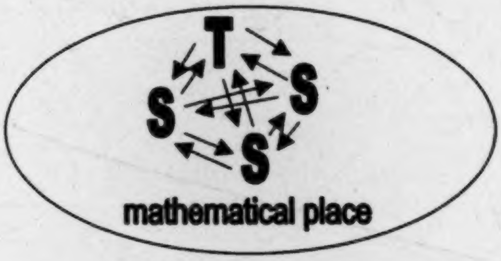


Figure 5. Neighbourly immersion

This model suggests that students and teachers occupy similar roles. If the letter *T* and any *S* were to be exchanged, the diagram would be the same. Students pay attention to their teachers' engagement with the mathematical place, and the teacher pays attention to the students' mathematical exploration. Although each participant pays attention to all the others, their roles may differ. The teacher's greater experience in mathematical places would skew the balance of mathematical authority. Furthermore, students know that they are placed under the teacher's social authority. I wonder if it is possible or wise for a teacher to relinquish these forms of power completely. Jaworski's (1994) Ben tries to temporarily relinquish only his mathematical authority during his students' investigative work. The kind of relationship described in this diagram does not seem to appear in my study, but I am nevertheless intrigued by its possibilities.

I characterize the more traditional transmission model of classroom dynamics with Figure 6. I use a train track because of the imagery evoked by Alrø and Skovsmose (1996) who describe the basis of traditional mathematics pedagogy as the ideology of certainty. Teachers in this model position themselves as guides who lead their students "on the right track". In this model, the teacher is like a step-by-step guide, moving students along a safe path. When a train engine teacher moves students along such a track, the students cannot see where they are going because the teacher blocks their view. The track is laid by experts and the teachers are "trained" by experts. Mathematics appears to be a hostile place in which only experts can dwell.

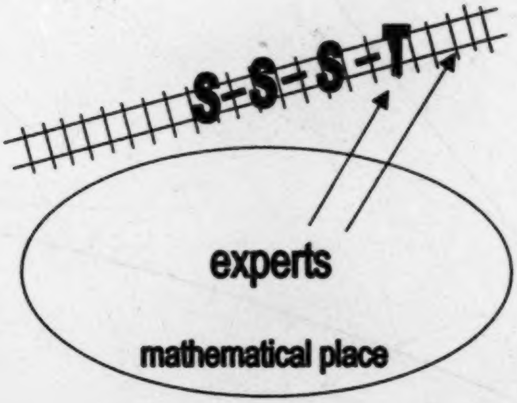
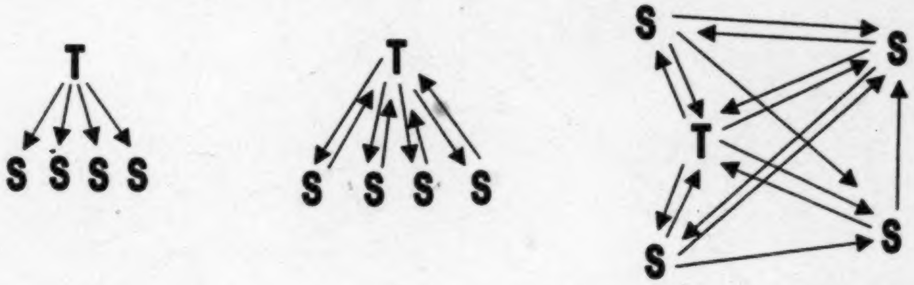


Figure 6. Transmission model (tourism)

There is one other classroom dynamic that I want to include with this set. My diagrams are inspired by Goldsmith and Shifter's (1997) diagrams. They, like Forman, *et al.* (1998), characterize their ideal teacher as one who orchestrates classroom discourse. Figure 7 is a copy of their representation of three classroom dynamics. They suggest that their third diagram describes the teacher-orchestrator. This diagram seems to resemble my neighbourly-immersion model. I wonder what in the diagram sets the teacher apart from the students. Orchestrators are not on the same level as their orchestras. Unlike the conductor, the orchestrator is not even necessarily present at the performance.



Goldsmith and Shifter, 1997, p. 31

Figure 7. Goldsmith and Shifter's (1997) models of classroom dynamics

I think that Goldsmith and Shifter intend a model more like the one described by Jaworski's Ben. The teacher helps students to pay attention to each other in mathematical exploration. If I interpret them correctly, then the teacher is in a different space from the students even though they occupy the same physical space. The teacher is in a management space and the students are in a mathematical space.

With her teaching triad model of a teacher's management of learning, Jaworski (1994) separates these two spaces but notes how they cannot be completely separated. Figure 8 is a representation of her triad based on the one drawn by her cooperating teacher, Ben (p. 144).

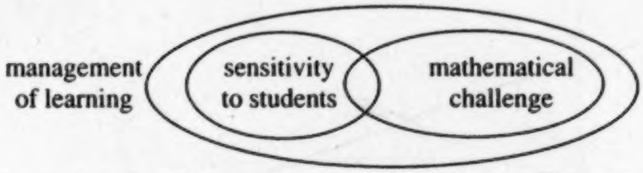


Figure 8. A representation of Jaworski's (1994) teaching triad

Figure 9 illustrates my synthesis of Goldsmith and Shifter's (1997) orchestration model and Jaworski's (1994) triad.

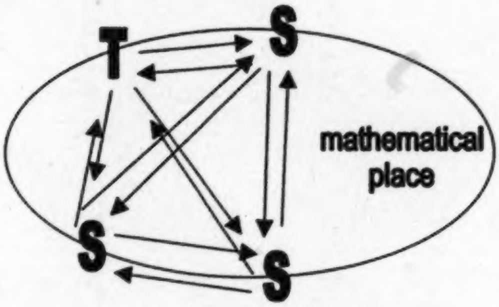


Figure 9. My way of seeing "orchestration"

Here, the teacher is controlling student interaction in the mathematical place, but is not personally involved in the place. Or, at least, the teacher is pretending not to be attentive to the mathematics in the place. They all straddle the edges of the mathematical place. Students are attending to teacher control as much as to the mathematics. And, the teacher cannot control the students' mathematical interaction without paying attention to the mathematics.

A Basis for Interpretation

The thoughts I share in this chapter form the basis of my interpretation of the audio and videotapes, and of notes and recollections taken from my field research. I am not as interested as I once was in evaluating the extent to which the students' experiences with investigative projects allowed them glimpses of the must-see curriculum sights or whether they seemed to be diverted onto a new and better mathematical track. I see mathematical experiences as complex places, part of interlacing webs of spatial, temporal, topical and mathematical relationships. In my interpretation of responses to mathematical problems, I look for evidence that might indicate how participants in the mathematical places position themselves. I am interested in how the problems become real to them and how they respond to this reality.

In the next two chapters, I interpret some of the classroom events using this new way of seeing. In Chapter 5, I focus on two particular segments of discourse and, in Chapter 6, I use these segments and others to focus on emergent themes.

Chapter 5 – Two Instances of Brief Immersion in a Mathematical Place

Before discussing the themes that I find common in the experiences of the students and teachers in this research, I describe, in this chapter, a pair of captivating classroom interactions. The first scene uncovers some of the inherent tensions associated with a particular teacher's way of positioning himself in the mathematics environment surrounding one group of his students. The second reveals tensions that a particular student feels with his brief immersion in a mathematical place. In both cases, the relationship between teacher and student is important, so it becomes necessary to consider the counterpart of each primary character.

Scene One – Mr. Penner and Natalie

The selection of transcripts in the first half of this chapter focuses on Mr. Penner's interactions with one of his student groups that worked on a response to the first of two investigative projects, entitled "Playing with Squares" (see Figure 2 on page 36).

Mr. Penner is an experienced teacher. Among his colleagues, he is looked to as a leader in both mathematics teaching and in school administration. His grade 10 class met for one 66 minute block each weekday, except for Fridays when they met for two back-to-back 66 minute blocks. He arranged to have his class do the two investigative projects in the second block on two Fridays with a two-week space in between. On each of these project days, at the end of the first block he asked his students to divide themselves and their desks into groups of three or four so that they could be ready to begin their tasks at the outset of the next period.

The scoring rubric for this investigation was distributed to the students in the first block on the Friday of the investigation I consider here. The rubric favours creativity, evidence of understanding mathematical concepts and processes, and clear communication that demonstrates mathematical thinking. Creativity is described in the rubric as an approach that "inspires further thought and exploration". The rubric is provided in Appendix 1.

Mr. Penner allowed his students to choose their own groups for working on the investigative projects. Before the project work began, he told me that "kids with common abilities hang [associate] together", so by choosing to let the groups self-select, he tacitly opted for groups in which students were likely to have similar levels of school performance.

Natalie, Kathy and Teresa were low achievers, all of whom eventually failed this Pure Mathematics 10 course. Preceding the first dialogue excerpt, these three girls had been concentrating on finding the exact heights of the squares. Their ideas, so far, had not satisfied them. Not recalling their recent class work manipulating numbers in radical form, they were square rooting the areas and struggling to find exact numerical answers. After twelve minutes, Natalie suggested to her friends a ratio-based approach to finding similar stacks of squares. It should be noted that this was not her first "Oh, wow!" (For this set of excerpts I include turn numbers. Turn 1 marks the beginning of their work on the project, so Natalie's exclamation is the ninety-ninth turn in this group's dialogue.)

99 Natalie: Oh, wow!
100 Teresa: Way to ...
101 Natalie: But don't we need to know the ratio of it? Like 20 to 45 is the ratio. If this was 72 what would that be?

102 Teresa: Well, that would be two-thirds more. So, what's two-thirds of 72? [*asking Kathy to make this calculation*] What's two-thirds of 72? So do 72 divided by three.
103 Natalie: What?
104 Teresa: 72 divided by three.
105 Natalie: [Mr.] Penner, can I have another calculator? [*calling Mr. Penner who is on the other side of the room*]

Natalie seems to have been *consumed* by the problem at this point. The Oxford English Dictionary (OED, 1989) describes *consume* this way: "to take up completely, make away with, eat up, devour". Natalie was completely taken up with her new insight. Although she and her friends were all participants in the mathematical place of inquiry, Natalie, for one reason or another, had become the most engaged at this particular time. Because her attention was being devoured by the problem, she began to see her friends in terms of their usefulness to her – for instance, asking them to perform the calculations she required (turn 101). Teresa and Kathy did not follow Natalie's rush of reasoning and were unable to perform the calculation Natalie wanted. Natalie was too excited to explain to them which numbers she wanted to compare using ratios. Natalie then called out to her teacher, asking to borrow a calculator that she could control. I infer that she expected that with her own calculator in hand she could work feverishly, independently of her friends. Kathy, in the meantime, continued following Teresa's instruction, seventy-two divided by three:

106 Kathy: 24. 24.
107 Natalie: So, what's that. Okay, this was 72… [*she seems to be talking to herself, ignoring Kathy and Teresa*]
108 Kathy: We're getting there. [*reporting to Mr. Penner who has just arrived with the requested calculator*]
109 Natalie: Oh, I get it. This would be 48. That would be 72. Right?
110 Mr. Penner: Absolutely.
111 Natalie: Are you sure?
112 Mr. Penner: Well, no.
113 Natalie: Yeah. That's right [*assertive*].

Okay. I have a question. What we're trying to do right here, right? You find the area and all the lines, right. But instead of 45 we're finding 72, right?
114 Mr. Penner: Well, sort of.
115 Natalie: But, don't we need to know the ratio between 20 and 45, and then if this is 72 what would be the ratio then?
116 Mr. Penner: Let me show you one thing.
If I wanted to find that [*pointing to the side of the 45-square*]
117 Natalie: Uh-huh.
118 Mr. Penner: What would I do?
119 Natalie: What do you mean?
120 Mr. Penner: What would I do? With that number?
121 Teresa: Square it.
122 Mr. Penner: Keep going with that. Close.
123 Teresa: Square root it?
124 Mr. Penner: Yeah. Say that.

Mr. Penner took control, and directed the group through a series of questions that Ainley (1987) might call "guess-what's-in-my-mind" questions (turns 118 and 120). After this portion of the interaction, he began to lead them through a comparison of the heights using mixed radical representations: $\sqrt{45} = 3\sqrt{5}$. Natalie was confused (turn 119) by his diversion. She might have been wondering why he was pointing to lines when she was looking at areas. Natalie incorrectly assumed that her teacher was interested in listening to her idea. Instead of trying to interpret her question or participate with the group in her reasoning, he listened in a way that Davis (1997) would classify as *evaluative* – "listening *for* something in particular … rather than listening *to* the speaker" (p. 359).

We might wonder why Natalie did not rebel when Mr. Penner usurped control from her. Alrø and Skovsmose (1998) describe a similar situation in a class they observed: "the teacher has one idea or one intention that he pursues, while the students have to guess their way" (p. 45). They suggest, in their case, that students "know that if they follow the teacher all will be well" (p. 44). When confronted with a choice between exploration and a sure route it is easier to choose the safe path. Morgan (1998) notes that when students write in response to a task assigned by a teacher, the safest path can be expected to be the teacher's path because "it seems likely that students will still perceive their primary audience to be the teacher" (p. 46). Natalie's group was now "safely" following Mr. Penner's reasoning. After more of Mr. Penner's guiding, the dialogue continued:

136 Mr. Penner: Well, is there another option you can use to find the value of that without using a calc- … like without using decimals? Have you ever seen that before? Have you ever seen that question before?
137 Natalie: Probably.
138 Mr. Penner: Yeah. Where did you see that?
139 Natalie: [*quiet for a while*] Okay.
140 Mr. Penner: What's that?
141 Natalie: Oh. Then you get three.
142 Mr. Penner: What's that right there?
143 Kathy: Three square root five.
144 Mr. Penner: Do you agree?
145 Natalie: I totally get it now.
146 Mr. Penner: Now tell me what that height is there.
147 Natalie: It's the same thing.

Natalie eventually concluded, with a convincing tone, that she totally understood it now (turn 145), and demonstrated, in the next few minutes of dialogue, that she was indeed able to follow Mr. Penner's approach to the problem independently. Yet, her group did not include any significant reference to this approach in their final report and she completely lost her ratio-based idea. Figure 10 shows a copy of their final report. The feature of their report that I find most striking is that it shows little of the richness of their mathematical thinking. In Chapter 6, I look at student writing in the context of the two investigations.

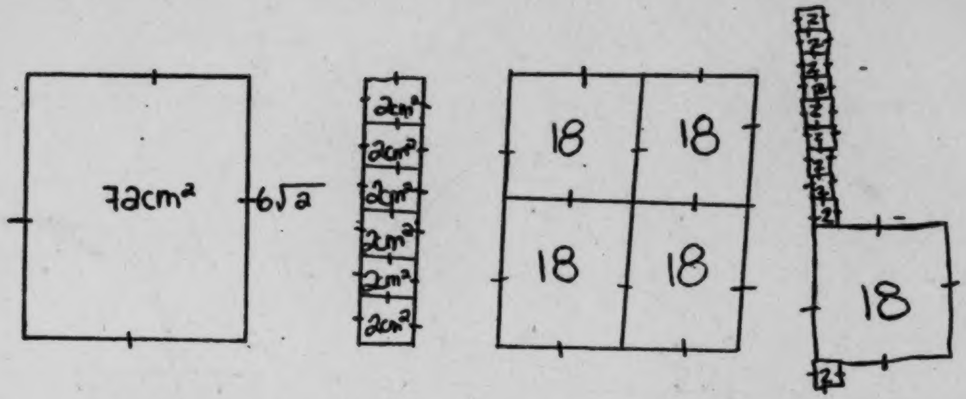


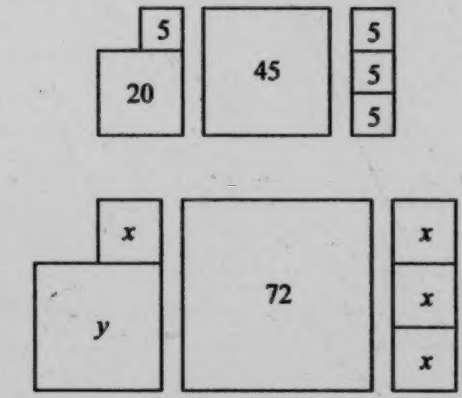
Figure 10. Natalie, Kathy and Teresa's report on "Playing with Squares"

Mr. Penner's Way of Guiding

Mr. Penner's allocation of class time for this open-ended task indicated that he was interested in providing open space for his students to explore collaboratively. On the day before the task, he expressed anxiety about the role that he would assume during his students' work on the investigative projects. He asked me for guidance as to what role he should play. I did not want to answer him directly because I wanted to observe what he would do without my influence. Instead, I described for him Mrs. Foster's reflections after her class had completed the same project. She had originally planned to tell her students nothing, but after the project she expressed regret for the amount of direction she had in fact given them. She noted that the groups she did not direct came up with the most creative approaches to the problem.

Mr. Penner said that he sympathized with her feelings and hoped to let his students work independently. But, like his colleague, in the moment of classroom activity Mr. Penner felt compelled to give hints to some groups. When Watson (1986) discusses tensions felt by British teachers conducting investigations she observes that "self-doubt is an ideal state of mind" (p. 16), because it demonstrates sensitivity to the many concerns related to investigative work. With Watson's wisdom in mind, Mr. Penner's hesitation seems to speak well for his approach to his students in this experience. His lack of assurance about how to position himself in this mathematical space demonstrated his sensitivity both to his students' feelings of productivity and to their need for independence. When asked after the project-work day how he chose which groups to help, he said that he had helped the ones that were struggling and getting nowhere. In Chapter 6, I present some of the creative responses submitted by the other groups – groups with which he did not intervene.

With his "evaluative listening" ears in place, Mr. Penner did not hear his expected approach to the problem and decided Natalie, Kathy and Theresa needed help. When I listen to this dialogue on audiotape, outside the constraints of the finite time and space of the classroom, I realize that Natalie's ratio-approach is viable. Figure 11 outlines my explanatory narrative for the continuation of Natalie's reasoning.



$$\frac{5}{x}=\frac{45}{72}=\frac{20}{y}$$

$$\swarrow \qquad \searrow$$

$$\frac{5}{x}=\frac{45}{72} \qquad \frac{45}{72}=\frac{20}{y}$$

$$360=45x \qquad 45y=1440$$

$$8=x \qquad y=32$$

Figure 11. An explanation of Natalie's ratio-based trajectory

Because the approach was new to me, I first doubted its possibilities. It seems likely that this group's poor performance on earlier tests led Mr. Penner to expect struggles in this group – to expect from them no insights that would be new to him – or that all insights would be wrong. Alrø and Skovsmose (1996) might suggest that Mr. Penner steered his students onto what he believed was "the right track", because he did not feel it necessary to listen to their "good reasons". He probably doubted that they could have "good reasons" without external help.

Whatever the reason for Mr. Penner's decision to "help" this group of students, his struggle between intervention and allowing the students to construct their own approaches is not uncommon. Williams and Baxter (1996) describe the struggle this way: "the teacher's dilemma is to have to inculcate knowledge while apparently eliciting it" (p. 24). Ball and Wilson (1996) agree:

> Intellectual honesty implies engaging students in the conjecturing, investigating, and argument that is characteristic of a field. But responsibility to students means grappling with the consequences of students reaching conclusions that their next teacher will see as wrong. (p. 182)

The Alberta curriculum outcomes suggest very particular approaches to some "types" of problems. Even the National Council of Teachers of Mathematics (NCTM), which asserts that the effective teacher knows "how to support students without taking over the process of thinking for them" (NCTM, 2000, p. 4), admits in its guiding principles the need to "focus on important mathematics – mathematics that is worth the time and attention of students" (p. 4). In this case, Mr. Penner seemed to feel that he knew which ideas were important and worthy of his students' attention, and he tried to elicit the kind of understanding that would prepare them for their next mathematics course.

Mr. Penner's way of being was not constant throughout this short interaction with Natalie and her friends. With his own unfamiliarity in this slightly foreign mathematical place, he shifted his position to find a comfortable niche in which he could be helpful to his students. First, he assumed the role of what I call an outfitter guide when he sent his students into the mathematical place opened up by "Playing with Squares". They became immersed in the problems associated with their new environment. They were no longer tourists being told by their guide what was important to know and what to do in given circumstances. A few of the groups received absolutely no hints or help from him.

As demonstrated in the given excerpts, the three students in one particular group varied in their depth of engagement with the problems of mathematics, but they all seemed more engaged than they were on the more typical days I observed. Later in this chapter I describe in more detail the depth of their engagement.

Apart from this scrutinized instance, in which Mr. Penner diverted the group from a creative possibility, his outfitter role seems to have been quite productive. Other groups, about which I report in Chapter 6, noticed mathematical possibilities that were new to me with absolutely no intervention from Mr. Penner. Natalie, Kathy and Theresa also displayed higher-level mathematical processes early in the project time. They were communicating, conjecturing and testing their conjectures. These are processes that were not explicitly part of their routine mathematical experience as I observed it in class sessions that were not project-focused. They had begun to discover a creative and promising approach to the task set before them – an approach that no other group in this research employed.

The second excerpt from this set revealed a problem with Mr. Penner's assumed role. The problem was sparked by his concern for this group; he did not have confidence in them. Mason (1988) lists confidence as one of the tensions for teachers who use investigations.

Fennema, Franke, Carpenter and Carey (1993) describe Mrs. J, a model teacher trained in Cognitively Guided Instruction: "Mrs. J. often said to [her students] that there were no problems that were too hard to solve, just some that they couldn't solve that day" (pp. 568-569). Mrs. J. displays the confidence necessary to allow her students to investigate independently. Because some problems are overwhelming for particular people at particular times, it was understandably difficult for Mr. Penner to be certain of these three students' readiness to live with this problem without help. Confidence in each group's readiness is an important requirement for outfitter guiding.

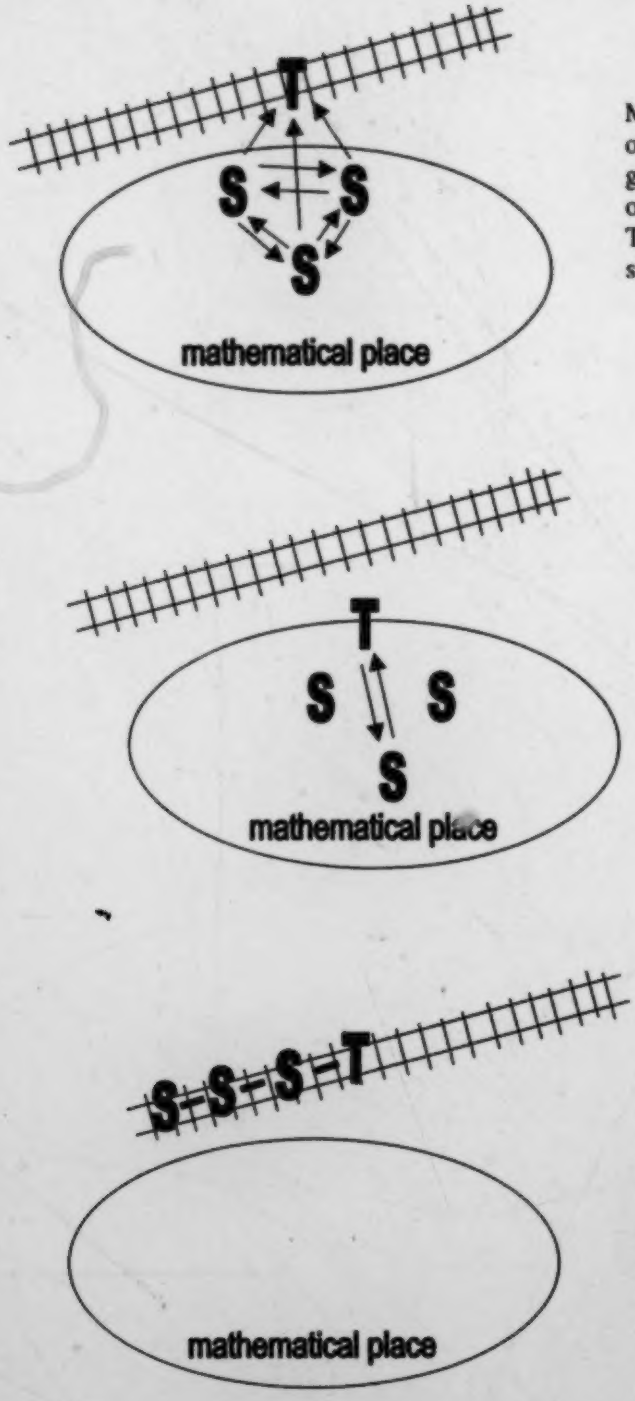
Although I balk at Mrs. J's suggestion that all problems are solvable, I support the way she encourages her students. Recalling Borasi's (1992) comparison of real-life problems to good mathematical problems, I have thought about the significant characteristics of real-life problems. For me to have a real problem there can be no apparent simple solution. If there is a simple solution there is no problem. Similarly, good mathematics problems ought to have no obvious or prefabricated solution available to the solvers.

When I have entered new domains of work, or new relationships, I have been supported by expressions of confidence in my adequacy. Those who have encouraged me – people who have outfitted me – have often been realistic about the likelihood that I would experience problems that would be beyond my power to solve. In the face of such problems, they have expressed confidence in my readiness to live with and within the problem and perhaps address aspects of the larger problem. Thus, Mr. Penner's way of being – his indecision about this group's readiness to bear the problems they experience – seems more realistic to me than the simplistic faith expressed by Mrs. J.

Figure 12 displays a series of images that relate Mr. Penner's interaction with Natalie, Kathy and Theresa to my models in Chapter 4. At first, the students were engaged in mathematical communication because their teacher nudged them into a new mathematical space. Because the teacher initially avoided participation in the new community that he set up for his students, his sudden, unsolicited advice confounded their mathematical conjecturing, testing and discovering. Because of the weight of his advice, fattened up by months of him being the authority, these girls were diverted from their engagement with the mathematical problem as they had been experiencing it.

After I played for Mr. Penner a brief segment of audiotape from the interactions that I describe in this chapter, he immediately replied, "I think this is where I help them too much." As his response to this audio segment continued, I sensed a struggle within him as he wondered, like I am wondering, what might have happened if he were to have not intervened:

> I could see they really weren't getting the idea, so I gave them an example. There was an example on the page, but I gave them an example more directly, with the roots. So, I kind of opened up that whole subject to them, and then I think they kind of got a few after. So, I mean I really led them into it, but that group is a group that struggles in math. Out of the three people, there's only one that's passing. I guess I could have left them alone and they would never have got it. Possibly. But, I think, was that the group that came up with some other creative method? I think they did actually, so maybe if I wouldn't have helped them along they would have come up with something.



Mr. Penner begins the project as an outfitter guide. He asks Natalie's group to report on its progress. He is outside the mathematical space. There is no "right track" for the students.

After not hearing what he was listening for, he begins to direct their attention to his way of seeing. At the same time his students begin to draw him into their mathematical space.

His way of seeing the problem is not based on his experience of immersion in their mathematical place. Instead, he focuses on his memories of the last time he was in a similar place. Thus he has one right answer in mind.

Mr. Penner suddenly moves back into the role he is accustomed to in mathematics teaching – guiding his students along "the right track". And the students jump right on. Natalie does not rebel.

Figure 12. Mr. Penner's ways of guiding in one instance

The other creative method to which he was referring was a sketch in this group's work that Mr. Penner saw as having possibilities. The extent to which he was reading in his own ideas is impossible for him or me to ascertain, because there was no evidence in this group of more thought or discussion related to the sketch.

After discussing with him his intervention with Natalie's group, I asked if he would change his level of intervention for the next project. He was not sure. He said that he thought he would "play it by ear."

Another way to look at Mr. Penner's relationship to the mathematics and the students in this mathematical place is to consider the language he used. When he first approached this group, he was looking into their mathematical space from the outside. He might have been trying to make sense of their activity in the place, but he could only relate their work to his own experience with a similar place – in terms of his way of adding numbers represented as radicals. His experience with the problem seemed to be limited to an approach that depends on finding radical representations for the heights of the squares, and using simplification and addition algorithms to manipulate these symbolic representations. Chapter 6 shows a number of samples of student work; Figure 21 (on page 120) seems to fit Mr. Penner's expected approach. Without listening carefully to Natalie, he could not assimilate her approach. It was "foreign" to his experience.

He did not spend the time necessary to understand Natalie's question. Instead, he seemed to be more interested in encouragement – a characteristic of outfitter guiding. He encouraged Natalie with unequivocal agreement, saying "Absolutely" (turn 110), but was thrown off balance in this position by Natalie's "Are you sure?" (turn 111). Natalie also seemed to have been diverted from her previous tenacious engagement, perhaps by his excessive confidence. They were both diverted from their intended roles. I consider the role Natalie intended for herself and her group later in this chapter.

109 Natalie: Oh, I get it. This would be 48. That would be 72. Right?
110 Mr. Penner: Absolutely.
111 Natalie: Are you sure?
112 Mr. Penner: Well, no.
113 Natalie: Yeah. That's right [*assertive*].
Okay. I have a question. What we're trying to do right here, right? You find the area and all the lines, right. But instead of 45 we're finding 72, right?
114 Mr. Penner: Well, sort of.

This brief exchange seems odd to me. If not for the diversion that each provided for the other, it would seem as though both participants were more focused on saying what they needed to say than on engaging in conversation. Mr. Penner wanted to express confidence, and Natalie wanted a passive audience for her exciting idea. Both of them were distracted from their aims by the other person's presence and means of engagement.

Shaken out of the detachment characteristic of the outfitter–guide, Mr. Penner was drawn into this group's mathematical space. He changed his pronoun use with his changed position, and began using the first person singular *I*. This is a pronoun he often used when demonstrating examples in routine mathematics classes. Natalie was drawn into the new positioning and changed her pronoun use as well, employing the second person pronoun *you*. Before this she had been using the first person plural *we*, presumably to position herself in an inextricable connection to all the people immersed in the problem space, including her groupmates and her teacher. At this point, the intermediate diagram in Figure 12 describes their relationship.

115 Natalie: But, don't we need to know the ratio between 20 and 45, and then if this is 72 what would be the ratio then?
116 Mr. Penner: Let me show you one thing.
If I wanted to find that.
117 Natalie: Uh-huh.
118 Mr. Penner: What would I do?
119 Natalie: What do you mean?
120 Mr. Penner: What would I do? With that number?

I am unsure whether Natalie was using *you* to refer to Mr. Penner here, or in the generic sense described by Rowland (2000) – where *you* is used instead of the pronoun *one*. Rowland finds that, when students shift from first person pronouns to the pronoun *you*, the change of pronoun sense often "signifies reference to a mathematical generalization" (p. 113). In any case, Mr. Penner soon moved toward a new position, the final position in my diagram sequence. He began to mask his personal voice by taking on the position of mathematical authority, speaking for the mathematical community.

He moved himself onto the track he perceived to have been laid by mathematicians before him and his students soon followed. Exploration of the mathematical landscape is severely limited from the perspective of a train moving along a track. There is one path through the territory. On "the right track", Mr. Penner no longer used the first person, but rather directed the group with imperatives and what Ainley (1988) calls testing questions, making sure they were following his procedures. The pronoun use in this excerpt from their dialogue shows Mr. Penner's leadership and his students' close following along "the right track".

120 Mr. Penner: What would I do? With that number?
121 Teresa: Square it.
122 Mr. Penner: Keep going with that. Close.
123 Teresa: Square root it?
124 Mr. Penner: Yeah. Say that.
125 Natalie: Square … okay.
126 Mr. Penner: Yeah. Do you agree – that whatever that is would give you that?

127 Natalie: Oh yeah.
128 Mr. Penner: And it would be the same as that.
129 Natalie: Okay.
130 Mr. Penner: Because it's a square – so all sides are the same, right?
131 Natalie: Yeah.

Mr. Penner shifted into the mathematical space that he initially avoided with this group and completely avoided with others. Perhaps the problems associated with his diverting this group from their potentially productive approach were rooted in his inability to be present the whole time in each group's mathematical space. Mr. Penner's problems are reminiscent of Shuller's (1983) description of her positioning during her students' investigative work in Britain: "I saw how difficult it was to be 'with' each student" (p. 38). Because he was circulating from group to group, Mr. Penner felt the need to make a quick judgment about what was happening in any group he visited. Wherever such understandably underinformed evaluations go awry, as one did here, he would be in danger of diverting students from their own productive mathematical thinking.

I wonder what would have happened if Mr. Penner had allowed himself to be immersed in sustained dialogue and discovery generated by this group– to participate as a neighbour in the mathematical place with these three students. What if Mr. Penner had entered into the problem world initiated by the task he had given to his students? Perhaps this would be the kind of situation Davis (1997) calls an inquiry environment, where the class members, including the teacher, seem "to be jointly exploring a mathematical issue rather than attempting to master already formulated bits of knowledge" (p. 368).

Although Mr. Penner was able to maintain his position as an outfitter guide to most of his students, with this particular group he demonstrated a source of possible tension with the outfitter–guide role. Either his concern for particular outcomes or his lack of confidence in particular students seemed to outweigh his desire to set the students free to explore a new mathematical territory. I wonder what difficulties and growth he would experience and inspire if he were to play the neighbour role in such mathematical terrain.

I conclude my comments on Mr. Penner's way of being, as drawn by this example, with this "what if" question. What if the teacher had immersed himself in his students' experience? It is too easy for me to speak authoritatively from outside the classroom culture about what Mr. Penner's best tactic ought to have been – as easy as it is for tourists rushing along Swazi roads to feel like they understand what they are seeing, feeling able to offer solutions to the problems they merely glimpse.

No. I do not want to position myself like the know-it-all expatriate aid worker described by a local Zimbabwean development worker: "Just imagine! We get back from a month's trip all around the villages and then some Dane, or German, or American tells us, No, the main thrust of the problem according to our information is…" (Lessing, 1992, p. 353).

As in any interaction within a web of complex relationships, there is no single right answer to the problems of the territory. Natalie told me after this particular mathematical experience that the investigative project provided time to look at one big problem and "it [gave] us a chance to use everything we know". Although I might see other possibilities for Mr. Penner in his guiding role with this group, he successfully created some space for these three students to explore and become consumed by the problems in a mathematical place. He stretched at least some of the cultural boundaries that would normally restrain these students' inquiry. Chapter 6 provides evidence that he provided even more exploration space for his students in other groups.

In the following interview excerpt, Mr. Penner described the value of the "Playing with Squares" project. In this excerpt, he seemed to be referring to the project instructions, but I suggest that his account describes well the whole experience he had set up for his students, including both the project instructions and his administration of the classroom environment.

> It did require [the students] to think outside of the box a little bit. Because, whenever I teach it's all so directed – you know, 'follow my pattern', 'follow my sequence'. Here, really, they had to do some thinking on their own.

Natalie's Way of Being

I suspect that Natalie felt as though her choices about the ways she might have positioned herself in her mathematics classroom space were limited by the authority figure her society had placed above her, her teacher. In the same way, Mr. Penner likely felt that his choices were limited by his understanding of society's expectations for him. Nonetheless, Natalie seemed to relish her perceived freedom to explore a new role in this mathematics classroom.

From the outset, she declared her intentions for how her group ought to position itself in this mathematical place. She positioned herself as a leader in her group. If I were to describe her as a guide, I would classify her as a neighbour to her two friends – a dominant neighbour. She began her group's investigation by reading aloud the investigation tasks, leading the group into territory opened up by the "Playing with Squares" task:

1 Natalie: [*reads the "Playing with Squares" task aloud*] What does that mean? Find stacks of squares that would be the exact same height as a square with an area of 72.
2 Kathy: Okay.
3 Natalie: I don't get what to do.
4 Teresa: Okay, 45 centimetres square…
5 Natalie: I do not get it.
6 Teresa: … is the exact same height as the two stacks…
7 Natalie: Let's try to work it out before we ask for help. Then we'll be a problem-solving group.

Natalie decided that she did not want help from outside. She preferred to solve the given problem with her partners and they did not argue against Natalie's expressed intention. It seemed that they correctly read their teacher's desire for them to immerse themselves in the mathematical space opened up for them by this task.

With this intention of working independently of their teacher's help, they positioned themselves in what I call a *zone of creativity*. My understanding of this zone is inspired by a recent documentary about Mennonite war veterans. This radio program quotes a young Mennonite who is part of a team that walks into conflict situations to avert violence without the use of force. He explains that as long as he allows himself the possibility of resorting to violence he closes the zone of creativity that would otherwise be there:

> When we do choose to keep violence in our back pocket as a last resort – to carry a weapon or to be willing to call in other people, police or military, to protect us using their violence – we cut off a creative channel that otherwise would remain open. The moment that I say in my head "if it comes down to it I am going to fight," that cuts off the energy to find another way around it. Because I don't believe that there's only two options, fight or run away. There's a whole bunch more in between. (quoted in Pauls, 2001)

In mathematics, there is a tendency to think that there is one right answer, implying that a response is either right or wrong. Yet there is more in between. The assumed right answer is a culturally-supported answer; right answers tend to be based on unarticulated common assumptions and arrived at following traditional approaches. For a teacher to evaluate students' responses to mathematics problems is to compare their work to the values that underpin the classroom culture – comparing to both the expressed and the assumed values. The word *value* is a root of the word *evaluation*.

Natalie and her group stepped into a zone of creativity similar to the one described by the young Mennonite. They decided that they would not resort to asking their teacher for help. However, the interaction described above demonstrates that, despite their intentions, "help" did come unasked, and when it came it shut them out of their zone of creativity and nipped the bud of Natalie's creative approach to the problem.

This group's willingness to enter into a zone of creativity probably shares a common source with their readiness to work on the task with undivided attention. I have found no indication as to what might have motivated them to such an extent. Natalie's disproportionate share of the turns in her group's dialogue and her aggressive body language throughout their engagement compared with the brevity and paucity of Kathy's talk and her laid-back posture suggest extremes of engagement. However, the difference might have been more a variance of personality. Kathy's comments were often insightful, suggesting that she was listening and evaluating all along, discerning the most important times to speak.

I was impressed by the focus this threesome maintained throughout the allotted time. If any one of the three had not been engaged, I suspect that the group would not have been able to stay on task to the extent that it did. Figure 13 counts this group's use of time during this investigation.

	Time on Task	Time off Task
chatting before reading the task		1 minute
engaged in the task	33 minutes	
distracted by a sneeze and a related joke		20 seconds
engaged	10½ minutes	
discussion about cross gender relationships		10 seconds
engaged	6 minutes	
after Mr. Penner's two minute warning they begin discussion about after school happenings		30 seconds
gathering their papers, deciding what to submit	1 minute	
chatting until Mr. Penner asks for the projects.		2½ minutes
Totals	**50½ minutes**	**4½ minutes**

Figure 13. Time use by Natalie, Teresa and Kathy

Eleven minutes of their 66 minute period was spent setting up before and cleaning up after the audio-taped portion of project work. Ignoring this group's one minute chat before they read their task, and also the time after their teacher's closure warning, the extent of their focus on mathematics is noteworthy. In this time period they were engaged in mathematics for 49 out of 50 minutes. I expect that any experienced mathematics teacher would be impressed, especially considering that these three girls eventually failed the course. They are certainly not the kind of students from whom I would expect such focus.

Although this group was deeply immersed in their project, they seemed to be unsure how to relate to one another in this new kind of mathematical space. Rowland (2000) looks at pronoun use to see how students and teachers position themselves with regard to each other and relative to the mathematics. These girls used a wide array of pronouns. Like Mr. Penner, they shifted around in this unfamiliar environment to find a comfortable and workable position. This short section of transcript exemplifies such jumping around:

81 Natalie: Twenty and five. If you add those areas together and then the square root … to find that it's five times five. And then you minus that five area? Do you understand what I'm saying?
82 Teresa: I kinda get it.
83 Natalie: I'm not sure you can do that but…we'll try.

Generally, they used the first person singular pronoun *I* to comment on their perceived state of understanding. For example, Theresa said "I kinda get it" (turn 82) and Natalie said "I'm not sure" (turn 83). Rowland (2000) notes that in mathematics classes the first person plural *we* often refers to the mathematical community, as in, "to multiply exponents we add the indices". These girls used first person plural when they described their group's action, but not to point to the mathematical community. They used the *you* pronouns to refer to generalizations, as in "If you add those areas" (turn 81), and sometimes to direct comments to specific members, as in "Do you understand what I'm saying" (turn 81).

Within this ten-second segment of dialogue these girls used personal pronouns in four different ways. Although this variety might reflect their disorientation in this open mathematical space, I cannot be sure about the senses in which they used their pronouns or their reasons for jumping around. Only the forms of their language are accessible.

Besides looking at voice, Rowland (2000) also looks at vague language in mathematics classrooms. He finds that "in the context of mathematical activity, uncertainty is a normal state, potentially a creative one" (p. 169). This creative potential approaches actualization as students and teachers become more aware of the space between the mathematical ideas that they are willing to assert and those that they actually believe. Rowland calls this space the Zone of Conjectural Neutrality (ZCN) (p. 141).

Generally, Natalie, Teresa and Kathy did not articulate their conjectures. Rather, they said out loud the numbers and operations that they were accustomed to writing in their books in normal classes. They did not explain the mental monologues that accompanied the symbols they wrote for their partners to see. When Mr. Penner arrived on the scene, the dynamic changed and Natalie tried to explain her conjecture. Thus, his mere presence coaxed out a part of her inner dialogue. She hedged, demonstrating her apprehension; she was inexperienced with mathematical explanation. Although Natalie was likely unaware of her shifting levels of confidence, she was moving about in the ZCN.

In the next excerpt, the language used by Natalie and Mr. Penner reveals how they changed their levels of confidence.

109 Natalie: Oh, I get it. This would be 48. That would be 72. Right?
110 Mr. Penner: Absolutely.
111 Natalie: Are you sure?
112 Mr. Penner: Well, no.
113 Natalie: Yeah. That's right.
Okay. I have a question. What we're trying to do right here, right? You find the area and all the lines, right. But instead of 45 we're finding 72, right?
114 Mr. Penner: Well, sort of.
115 Natalie: But, don't we need to know the ratio between 20 and 45, and then if this is 72 what would be the ratio then?
116 Mr. Penner: Let me show you one thing.
If I wanted to find that.
117 Natalie: Uh-huh.
118 Mr. Penner: What would I do?

Natalie often hedged her utterances by tacking the one-word question "Right?" at the end of her propositions, as in turn 109. Mr. Penner seemed uncomfortable with her hedging, and responded with the ultimate anti-hedge, "Absolutely". After Natalie questioned his confidence, he also moved into the ZCN by prefacing his evaluation with the word *well*. It feels for a moment, in turns 110 to 114, like more typical roles of the teacher and student are reversed. This teacher moved into the position of interpreting the student's mathematical explanation, and realized that he could not be sure that he understood her. Mr. Penner retrieved the reins of control in turn 116 and moved abruptly out of the ZCN into a place more familiar to him – a place where he would guide students along "the right track".

My interpretation of the interaction between Mr. Penner and Natalie's group focuses largely on the choices teachers have in positioning themselves during their students' investigative work. These three students' experience was significantly affected by their teacher's chosen role. Mr. Penner gave mixed messages. First, he immersed his students in a mathematical place, giving them a task that would focus their attention on the complexities and multitudinous possibilities there. Natalie then led her friends into a zone of creativity, but they were distracted by the authoritative intervention of their teacher and were subsequently moved out of that zone. They passively went along.

In this instance, the source of tension seems to lie between teacher and student. In the second half of this chapter, I consider a scene in which the struggle seems to be between a student and the mathematics, or at least between a particular student and his conception of mathematics – a struggle the teacher allowed.

Scene Two – Greg and Mrs. Foster

This selection of transcripts focuses on Greg's frustration with a kind of mathematics with which he was unfamiliar. After a positive experience with the first project, "Playing with Squares", he struggled throughout his group's work on "Parallel

Division" (see Figure 2 on page 36). Greg's teacher, Mrs. Foster, was an experienced high school science and mathematics teacher. All her experience had been in rural schools much like the school in which this scene was set.

Greg's grade 10 mathematics class met for 80 minutes each day. The day that was devoted to the "Parallel Division" investigation followed the earlier project by two weeks. Mrs. Foster assigned groups randomly for each project, so for the second project Greg was working with a different group from the one he had been with two weeks earlier. His second group was aware of his pivotal contributions in "Playing with Squares".

The scoring rubric that Mrs. Foster used for evaluating student work in both investigations is the same one that Mr. Penner used for "Playing with Squares" (see Appendix 1). Mrs. Foster had distributed the rubric the day before the first project day, and had explained the expectations for evaluations of excellence. The rubric favours creativity, evidence of understanding mathematical concepts and processes, and clear communication that reveals mathematical thinking.

Although I am particularly interested in Greg, I preface my discussion of his way of being with a description of his teacher. The way she positioned herself in Greg's mathematical space is an important part of the setting.

Mrs. Foster's Way of Guiding

The National Council of Teachers of Mathematics (NCTM, 2000) asserts, "Students learn more and better when they take control of their own learning" (p. 5). Teachers are thus encouraged to "support students without taking over the process of thinking for them" (p. 4). Although Mrs. Foster seemed to embrace this philosophy, her students did not always demonstrate control over their learning.

In my last discussion with her, more than a week after her class undertook the second project, I asked her to describe which mathematical tasks are most valuable for her students. She answered:

> I think that the curriculum in itself isn't really what's important. To me, that's what math is – problem solving – not knowing trig. identities and being able to get to the answer. It's the process and the thinking process. I always try to express that there's never just one way to do something.

She expressed concern before directing her students to attempt the first investigation, saying "I don't want to lead them to do what I did [when she did the project in preparation for the class]". After the project, she described for me the struggle she faced in allowing her students to experience the frustration of independence and growth:

> It was tough seeing them frustrated, and I don't like seeing them frustrated, but I think I would try and not give as much direction, just kind of give a few pointers. Because some of them I actually ended up steering. Even though that's not my intention, that is kind of what I did.

Mrs. Foster was trying to position herself to guide as an outfitter. She idealized herself outside of the mathematical place in which she immersed her students. Her resolve to remain external to their work was made clear in the way she prepared for the second project. She explained to me that she had intentionally avoided doing the project herself to prevent herself from guiding students the way she had done the previous time. In this second project, she felt that she could not steer her students because she had not yet ventured into a mathematical place like the one opened up by the students' task.

Greg's Way of Being

Greg's mathematical place was not only influenced by his teacher's positioning in it, but also by his experience from the earlier project. In my interview with his group for "Playing with Squares", he explained that his group work appears different from his usual mathematics work because, when working with a group, he perceives the need to explain his thinking so that the others understand well enough to notice his mistakes. He summarized his role in the first investigation in this way: "I came up with the formulas." The others agreed that Greg had been the primary source of their group's conjectures, yet they each had felt important in the group's work. They had tested Greg's ideas and noticed problems with his explanations. It is noteworthy that this group had hoped to work together again, even though its peer-conscious members included a mix of popular and unpopular students. Together, they had experienced a fulfilling mathematical place in their first investigation.

Mrs. Foster described Greg as a student who lacked self-esteem. His experience with this group changed his feeling of self-worth, at least for a time. Although Greg was well aware of the possibility of incorrect conjectures, he exuded extreme confidence in his group's result. His group's vigilance in testing their conjectures seemed to be the source of his self-assurance. This excerpt from my interview with his first group demonstrates his self-confidence.

> Wagner: What happened if someone disagreed with something one of you said?
>
> Justin: We let that person kind of say what they thought was wrong, and we'd check it out probably – go back and try to figure it out.
>
> Wagner: And it seems to me I heard that happen a few times. When someone said that you were wrong, did you feel bad about it at all?

Greg: No. Cause there's always the chance that you can be wrong.
Wagner: And because it was done respectfully?
Justin: We don't even know if we got the right answer.
Greg: We did.

Greg had demonstrated comfort in the Zone of Conjectural Neutrality (ZCN), but when looking retrospectively at his group's work he felt justified in moving beyond the ZCN because of his faith in their careful testing. His unequivocal "we did" is a vivid departure from the hedging that usually accompanies student language in the ZCN.

His experience with the first project prepared him with a feeling of security for immersing himself in the second project. With this investigation, he felt confident that in collaboration with his new group he would be able to find satisfying answers to their tasks. However, part of his experience with "Parallel Division" would change his attitude to such an extent that he would become reluctant to talk about his experience. After this second project, I asked "Did you like this one?" His response was "No – hated it". Later in this chapter, I use transcript selections to consider what could have made this mathematical space inhospitable to Greg.

The audiotape record of Greg's group's work is characterized by extended silent sections, in which it is apparent that the three students were working individually on paper and also watching each other work. From their comments on each others' results, I can only reconstruct likely unarticulated conjectures that they seem to have been testing.

After reading the instructions, they immediately carried out the division calculations from the given setting. Assuming they divided correctly, their results would have looked like Figure 14.

$$(2n^3+7n^2+8n+3)\div(2n+3) \qquad 2783\div 23$$

$$\begin{array}{rllll}
 & n^2 & +2n & +1 & \\ \cline{2-5}
2n+3\,\big) & 2n^3 & +7n^2 & +8n & +3 \\
 & 2n^3 & +3n^2 & & \\ \cline{2-3}
 & & 4n^2 & +8n & \\
 & & 4n^2 & +6n & \\ \cline{3-4}
 & & & 2n & +3 \\
 & & & 2n & +3 \\ \cline{4-5}
 & & & & 0
\end{array}
\qquad
\begin{array}{rcccc}
 & & 1 & 2 & 1 \\ \cline{2-5}
2\ 3\,\big) & 2 & 7 & 8 & 3 \\
 & 2 & 3 & & \\ \cline{2-3}
 & & 4 & 8 & \\
 & & 4 & 6 & \\ \cline{3-4}
 & & & 2 & 3 \\
 & & & 2 & 3 \\ \cline{4-5}
 & & & & 0
\end{array}$$

Figure 14. Calculating the quotients in "Parallel Division"

It is impossible to understand the exact source of Greg's frustrations, but a variety of evidence points to his astonishment with apparent inconsistencies. It seems that his group noticed that the coefficients throughout the polynomial division calculation were the same as the digits in the numeric calculation. The group members seemed to agree that this parallelism ought to be evident for any polynomial division.

Mrs. Foster checked on their progress about ten minutes into their work time, after they began testing their conjecture. Figure 15 shows a polynomial division calculation that would falsify their conjecture.

$$(6n^3+18n^2+7n+8)\div(3n+4)$$

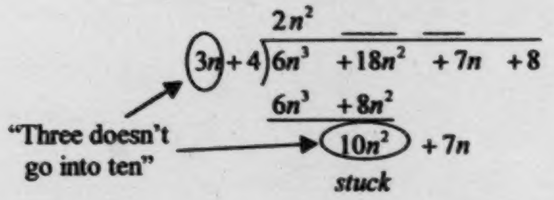


Figure 15. Testing the parallelism conjecture

I do not have access to their scrap paper, so I cannot know the actual calculation they used to falsify their conjecture. In Chapter 6, I discuss these and other students' failure to include in their written work the examples they used to test their conjectures. Judging by this group's discussion, they found their conjecture false using a calculation similar to this one.

Angela: I tried doing that – changing the numbers in here – and it doesn't work out. Three doesn't go into ten.
Mrs. Foster: Okay. Now you need to decide where to go from there. [*laughs a little*]
[*quiet for a bit*]
Michelle: Just say it doesn't work.
Mrs. Foster: Well, what do you think your next step is?
I can't tell you what to do. That's why we're doing this thing. It's for you to kind of self-direct yourselves. So, what do you think the next step is now?
Angela: Use different numbers?
Mrs. Foster: Okay. That's an option. Try it.
Angela: But if you change these numbers, do you have to change these numbers?
Mrs. Foster: That's up to you. [*laughs a little*]
You're getting very frustrated with my answers, aren't you? [*laughs a little*]
Greg: I hate this!

Mrs. Foster's "I can't tell you what to do" may seem like an exaggeration. She could have told them what to do, but from her chosen position as an outfitter guide, she could not tell them.

Greg seems not to say much when he is upset. After this exchange, the group persisted with trying to find an explanation for which statements would work. After a few minutes I happened upon the scene and they apprised me of their progress.

Greg:	We've only found so far one example that it actually follows what I thought would happen.
Angela:	I'm changing all the numbers and all the signs and all the different frigging things, and the little things right here, and it's still not working. It doesn't work!

Their words and tone revealed their emotion – a sign that they were captivated by the problem.

Before observing this experience, I had blindly promoted classrooms in which students would be immersed in mathematical spaces without teacher direction. Here, I became aware of a source of potential trouble in such places. The students were troubled because they had a problem. Kilpatrick (1987) asserts that people have to construct problems for themselves – they cannot receive them from someone else. In this sense, these particular students constructed a problem for themselves. Greg and his groupmates let the problem consume them; they felt emotionally invested in their ability to solve it.

Greg's emotion was directed at the problem in this mathematical place – not at external realities. Late in their work, Michelle reported to me about their group's mood and Greg clarified the object of his anger.

Michelle:	[*laughs*] Our whole group's stressed.
Wagner:	You're having troubles, eh?
Michelle:	It's going to be okay, Greg. We're going to pass it.
Wagner:	Is that what you're worried about, or is it just that you're annoyed?

Greg: I'm extremely annoyed. And, that's basically it. [*Michelle laughs and then Wagner laughs*]

Greg's mystification with this problem suggests that his prior experiences with mathematics had been dominated by examples that worked out nicely and tasks that did not push the boundaries of the given examples. We might blame his frustrations on the "Parallel Division" task. Indeed, I feel pain whenever I revisit Greg's transcripts. I feel responsible for his pain because I constructed the task that encouraged him into a place he found hostile.

Alternatively, I might blame his frustrations on the sum of his mathematical experiences. Why, after nine years of formal mathematics education, was Greg unable to cope with a mathematical setting that did not have a readily apparent explanation? He had been sheltered; he was naïve. Like a houseplant that is moved outside, he had revelled in the fresh air of the new and open space set before him two weeks earlier. With the second project, he was like the formerly-sheltered houseplant that is bent and broken when exposed even to gentle wind.

Responses to Greg's Frustrations

It is difficult for me to watch others suffer. With various exposures to other people's troubles, I have felt sympathetic pain, I have disturbed my world with "Why must this be so?" questions, and I have asked myself how I might intervene to avert future suffering. So far, each of these responses has been part of my response to Greg's apparently painful struggle. The above transcript excerpt reveals another kind of response of which I am not proud. I laughed.

Mrs. Foster and Michelle laughed too. Is laughter a bad response? It is too easy to speak authoritatively from outside the classroom culture about what a better response would have been. At the time, I felt bound by the territory. I sense that Mrs. Foster felt the same way. We both tried to encourage Greg without yanking him out of his mathematical space. In retrospect, I cannot imagine how we could have extricated him from this place, because he was consumed by the problem.

The following transcript excerpts describe the difficulty inherent in attempts to reach from the outside to someone immersed in a problem. About forty minutes into his group's work, Greg grasped at help:

Greg: Can you give us some sort of hint?
Mrs. Foster: Well, what's the problem?
Greg: I can't figure any of this out. I don't see what's supposed to be similar, or whatever.
Mrs. Foster: Nobody is saying there's supposed to be something similar. Maybe you're on the right track.
Greg: What?
Mrs. Foster: The question just asks, "Can your observations be generalized for all polynomial divisions?" So, you've tried other polynomial divisions.
Greg: Yeah.
Mrs. Foster: And did it work?
Greg: For some, not all of them.
Mrs. Foster: Good. There you go. That's your observation.
Which ones did it work for?
Greg: This one and this one.
Mrs. Foster: So, what types of things are maybe different?
Greg: I don't know.
Mrs. Foster: [*laughs*] You're on the right track.

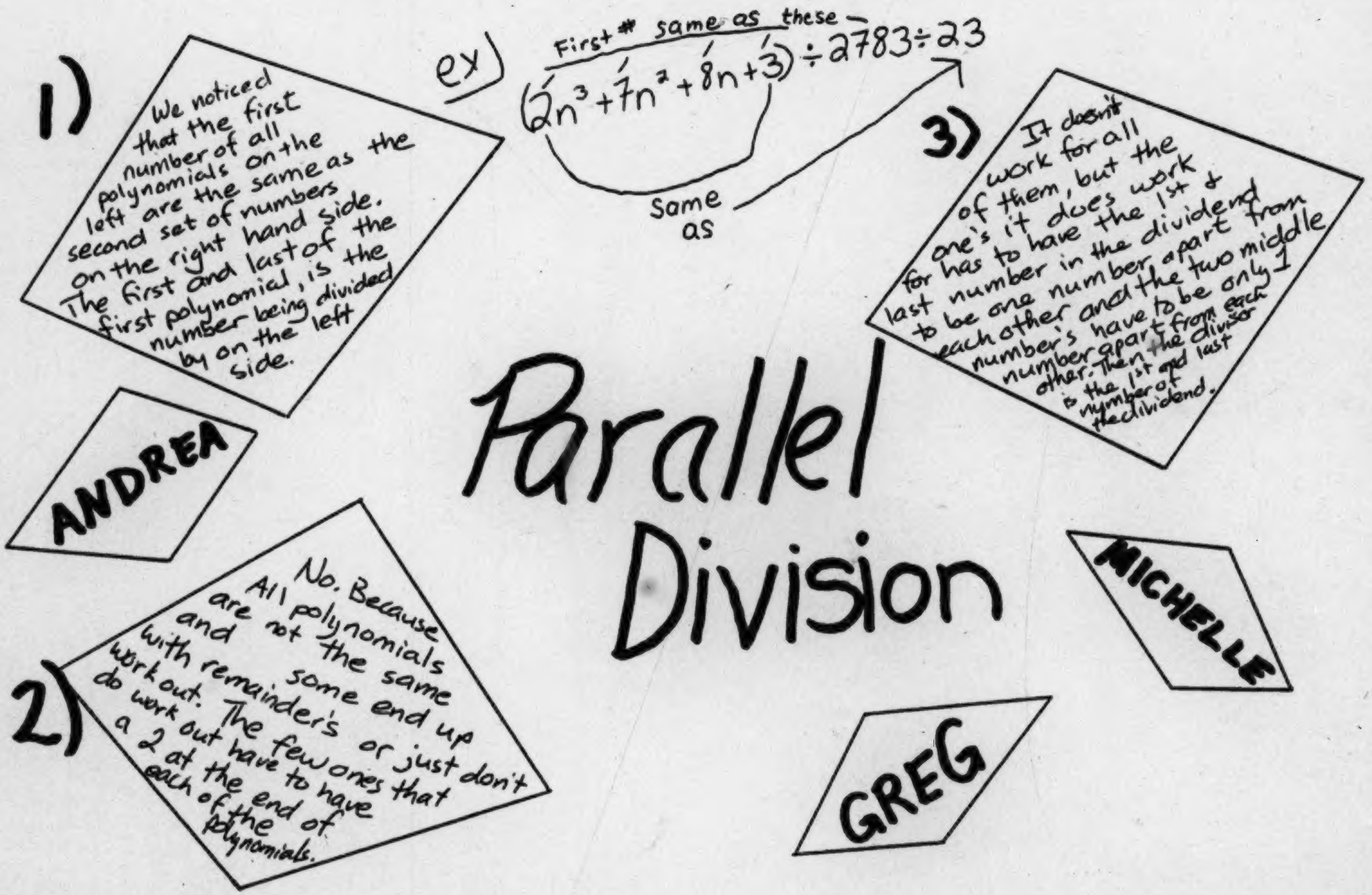
I played the audiotape of this excerpt for this group in my interview with them. Here is their response to it:

Wagner: What did Mrs. Foster mean when she said "You're on the right track"? What do you think?

[*quiet for a while... no response*]
No idea? Did you feel like you were on the right track?
Greg: [*answering immediately*] No!
Michelle: No.
Wagner: Why not?
Greg: Because every time we tried to do something different we always ended up in a dead end.
Wagner: What kind of dead end?
Greg: Well, like the thing we were trying to do didn't work.
Wagner: Oh, your idea that you thought you noticed?
Greg: Didn't work out.
Wagner: Does that mean that you were doing something wrong then?
Greg: [*immediate response*] Yup.

Later in the interview, Angela explained their interpretation of Mrs. Foster's "You're on the right track" encouragement: "She just did that to help us to feel better." It seems plausible that no matter what Mrs. Foster might have said or done, this group would have remained mired in frustration. Because of their lack of experience being immersed in mathematical territory, they were confounded by complexity. They did not realize that the mathematics they normally saw in class was a simplified version of other people's problem-fraught exploration.

Another reason for this group's frustration is that they did not value their own mathematical thinking. Mrs. Foster rated this group's work above all the others because they were making and testing conjectures. The students, however, felt that only simplistic explanations and conjectures were valuable in mathematics. Like the written work submitted by Natalie, Kathy and Teresa, this group's submission contains little evidence of their mathematical thinking. Figure 16 is a copy of their report (with the names changed).


Parallel Division
ANDREA
MICHELLE
GREG
ex)
First # same as these
$(2n^3+7n^2+8n+3)\div 2783\div 23$
same as

1) We noticed that the first number of all polynomials on the left are the same as the second set of numbers on the right hand side. The first and last of the first polynomial, is the number being divided by on the left side.

2) No. Because All polynomials are not the same and some end up with remainder's or just don't work out. The few ones that do work out have to have a 2 at the end of each of the polynomials.

3) It doesn't work for all of them, but the one's it does work for has to have the 1st & last number in the dividend to be one number apart from each other and the two middle number's have to be only 1 number apart from each other. Then the divisor is the 1st and last number of the dividend.

Figure 16. "Parallel Division" work by Greg, Michelle and Angela

In this scene, Greg was frustrated by his realization that mathematics was not as simple as he thought and by him wanting a step-by-step guide after all. His frustration was rooted in his lack of experience with mathematical exploration – an experiential void compounded by sudden immersion in a foreign space. Although his teacher tried to encourage him in his exploration, the depth of his connection to the space, perhaps in conjunction with her lack of participation in the space, rendered her encouragement ineffectual.

Summary

Connecting this scene to the first scene where Mr. Penner struggled with his position in the mathematical territory, I see opposites and similarities. Mrs. Foster remained firmly in her outfitting role and let frustration consume her students' experience, whereas Mr. Penner abandoned his outfitting role, perhaps to protect his students from frustration. All the players in these two scenes shared a lack of experience with mathematical immersion and suffered certain consequences of sudden exposure. In the next chapter, I include some of the other students and their interactions with their teachers as I outline some of the themes that presented themselves to me as characteristic to such experiences with sudden immersion in mathematical places.

Chapter 6 – Some Characteristics of Brief Immersion

In the previous chapter, I describe one scene from each of the classes that participated in this research. This chapter is organized by themes instead of scenes in order to provide a different perspective on the project work in the same two classes. I draw upon the wider body of student interactions and teacher interactions with students for examples that provide colour for my exploration of the themes that appear to be inherent in the project work done in both classes. I ask what are some characteristics of immersed participation in mathematical places.

When immersed I am captivated, or I might say consumed, by the problems of the new place because I become a participant in the community that experiences the problem. The problem can no longer be looked at "objectively" from outside the place for, with my move to be inside the place, the problem moves inside me. The problem is complex because the structure of the community and the limitations of each participant are part of the problem. However, as a connected participant, my rejection of simplistic responses opens for me a zone of creativity that cannot be entered without immersion in the place. Within this zone of possibility, participants' choices regarding their problems reveal their values.

This chapter includes a discussion of three characteristics of student investigation – captivation, complexity and creativity. I rely more heavily on the written submissions of participating students in this chapter than in Chapter 5. Indeed, the students' submitted reports provide glimpses of the values that underpin their choices. Therefore, a part of this chapter focuses on the values revealed in the students' writing.

Captivation

When my family and I lived in Swaziland, we could not escape the problems inherent in our community. The problems affected everything we did and everyone we loved. For instance, reliable access to clean water was a problem that every community member shared. At times we ignored the problem so that we could attend to other concerns, and at times our energies were consumed with finding partial solutions to the problem. We knew that the problem would not disappear; it could not be solved once and for all. Every community in the world struggles with the same problem in varying degrees.

We actually could have escaped this problem and our community by telling the organization that placed us there that we wanted to leave, but somehow we felt like we could not escape. Because we had chosen to immerse ourselves in the community, we would have had to change the way we perceived ourselves in order to abandon our new neighbours and escape the community's problems. There is a sense in which such abandonment would have felt like self-inflicted violence – such a choice would have been a violation of who we were, or who we wanted to be.

For me the most striking characteristic of the participant students' mathematical investigation was their overwhelming commitment to the given tasks, their captivation. Mr. Penner, too, was particularly impressed with the intensity of his students' work. In my interview with him after the first project he noticed that no student asked for a break during the project work. The absence of this request is significant because Mr. Penner regularly gave his students a break to stretch midway through each mathematics period –

they would take a one or two minute walk through the halls together, as a class. His students had come to expect such a break and often asked for it in routine classes.

In Chapter 5, I describe the focus of Natalie and her friends by counting their time on task – a significant measure considering the group's usual mathematical performance and the wealth of opportunity for distraction. Similar focus was the norm in both classrooms. Although the groups were typically engrossed in their investigative tasks, some groups held individuals who seemed less engaged than others. In both investigative projects, only one group out of fifteen did not become engaged with its tasks. This group remained disengaged from both investigations.

In Mr. Penner's class, Terry, Brian and Shawn exemplified the depth of their immersion with this short exchange 55 minutes into their work on the "Playing with Squares" task:

> Terry: We've got ten minutes to get the next three questions.
> Brian: There's more questions?

Time sped along outside of their awareness as they spent the first 50 minutes engrossed in one part of the project. The external expectation that they would have to do all the questions disengaged them from the problem that had captivated their attention. This group's written submission is shown later in this chapter (Figure 20 on page 116).

Even Mrs. Foster became captivated by one of the projects during class time. She had decided to not try "Parallel Division" in advance of her students' work on it as a way of preventing herself from steering them. Halfway through her students' work on the task she decided to work on it a bit. She became consumed by her conjecturing and testing and went off into a corner of the room to work at it. After a while she stopped herself and apologized to me for becoming disengaged from her students. Her eyes were filled with wonder after this time that she called "playing". She explained to me how all her conjectures proved false. She began to understand in a deeper way Greg's feelings of frustration. However, her experience was different from Greg's because she had had more previous experiences of immersion in complex mathematical spaces. Because of this prior experience, she was able to feel wonder instead of frustration.

Greg's captivation with the problem seemed to be making him sick. He could not extricate his sense of self worth from his perception of his success with the problem. In Chapter 5, I acknowledge Greg's captivation and ruminate about the source of his frustrations.

With the "Parallel Division" task, which propelled Greg into a seemingly inhospitable mathematical space, there is evidence from a number of groups of a stage beyond captivation – a stage in which some students distanced themselves from captive engagement. While some students worked feverishly, their partners did not always share their passion. Greg's partners, for example, were distracted from the mathematics by Greg's frustration. They poured much of their energy into consoling and encouraging him.

Aaron, who classified himself as lazy in mathematics class, was compelled to explore many new ideas while his partner and friend Brent resisted such fervour. Brent was satisfied with their group's one clear finding while Aaron continually repeated "We're missing something". In my interview with them, Aaron explained his atypical captivation:

> Aaron: [The project] takes your mind off stuff that you're doing in class. You need a break sometimes.
>
> Wagner: How many divisions [polynomial divisions] do you think you did during that class?

Aaron: I filled about 5 pages?
Wagner: And how many would you have done in a regular class, the kind of class that you need a break from?
Aaron: I wouldn't…
Wagner: Two pages?
Aaron: Not even. Just one.
Wagner: I hear you saying you need a break from the regular class when you're doing one page of division. Then you do five pages of division and you call it a break.
Aaron: Cause it's just one question.
Brent: It's not a break.
Wagner: So why was it a break then?
Brent: Not going to school. That's a break.
Aaron: I don't know. This is only one question. When you're doing stuff in school you have like twenty questions.
Wagner: Oh, so it's better to have one …
Aaron: Yea, and work on that one more.

Ironically, Brent was by far the more successful student of these two when compared in the conventional sense of success in school. Brent held one of the top two positions in the class standing but Aaron was barely passing. The following excerpt from their group's work on the project is typical of their interaction during the project. It came approximately 40 minutes into their work.

Mrs. Foster: Aaron, you have three pages of…
Aaron: Exactly. I'm trying to figure out …
Brent: I can start writing the words and stuff, right? Oh, I guess I can.
Aaron: Just wait a second.
Brent: You're doing too much work! [*exasperation*]
Aaron: I'm thinking! [*even more exasperation*]
Brent: Just skip the ones that are…
Aaron: Shut up.

While Aaron was consumed by the mathematical landscape opened up by their task, Brent was distracted by an external reality – the imminent evaluation of their work. Their written submission is shown later in this chapter (Figure 18 on page 110).

Brent wanted to put something on paper while Aaron was dissatisfied with their findings. Brent was immersed in the problem, especially at the beginning of their work on it, but for some reason he became more distanced from the problem that captivated Aaron. Perhaps he was attracted more to sharing their ideas than to exploring new ones. Or, with his higher class standing, he might have been more worried about their work's evaluation and thus more concerned with writing at least something down for submission.

Both approaches can be seen as consistent with participation in a complex community. In my experience, it is impossible to remain engrossed with a problem, even an important one, for extended periods of time without feeling depleted, and without neglecting other important expectations that are part of my participation in a community.

I suggest that for people immersed in a complex space, responses to its problems cannot be completely satisfying. Yes, Aaron's group verified some and falsified some of their conjectures, but these findings opened up new questions that Aaron felt compelled to pursue. That which was somewhat satisfying opened up new questions and concerns. This suggests a life-embracing way of being, where quenched thirst inspires new work and new work drives thirst.

I suggest that it is important for a teacher to support students in their participation in mathematical investigation, whether they are consumed by a problem or temporarily distanced. A problem that consumes a particular student might be seen as an opportunity to engage other students in looking at their mathematics from a new perspective and to seed fruitful class dialogue.

Complexity

In Chapter 4, I describe the nature of problems in complex places. I use Swaziland's problem with AIDS to compare the simplistic ease with which someone living outside the place can express "the" answer to the problem with the complexity of the problem for participants in the community. A shallow tourist's view of a problem allows the tourist to see only one part of a problem. A particular answer may seem obvious in this case. For participants who see the interconnectedness of all the problems in their space, no single answer satisfies.

Aaron's fixation on this one question attests to its depth. Aaron compared his experience with this question to what he called normal mathematics classes in which he would be told to answer many questions similar to one another. Part of the richness of this particular question comes from the way it is phrased. It is open-ended. Another part of its depth was opened up by the amount of time allotted to it. Because Mrs. Foster allocated a significant amount of time to it, students were tacitly led to understand that it would not be very simple.

No, the task is not very simple at all. The interconnectedness of algebraic operations and whole-number operations is fundamental to conventional algebraic manipulation. Algebraic expressions have much more potential for ambiguity and flexibility than whole-number expressions have. Because of this potential, algebraic algorithms based on whole-number operations are unlikely to be conveniently consistent.

Open-ended tasks, like the investigations addressed by these two classes, demand mathematical conjecture. When we test conjectures, we either verify them or falsify them. However, there are not only two possibilities. There is much in between. Most untrue conjectures come from particular true cases, and conjectures might well be true for certain other cases. The task of finding the cases for which such conjectures are true may not be simple, but it is a task that can open up rich mathematical territory.

Greg and his group apparently conjectured that any polynomial division has a parallel numeric division with coefficients matching digits in the question, the calculations and the quotient. Nearly every group in both classes made this conjecture at some point. Another group called the numeric division the mirror of the polynomial division. Greg's group found this conjecture untrue; they falsified it with one single example early in their work. Greg was confounded by the example given in the task, the only one that fit their conjecture in the early stages of their work. In their frenzy to consider different cases, they found some polynomial expressions that seemed to fit their conjecture and some that falsified it. The problem, they realized, was with the simplicity of their conjecture. They needed to explore to find a conjecture that explains both the expressions that have mirrors and those that do not. Greg and his group seem to have made more specific conjectures and found each of these too simplistic.

Unfortunately, the students in Greg's group and most of the other groups were not acculturated to value such high-level mathematical thinking. They were accustomed to looking for one simple right answer, which could be checked in the answer section at the back of their textbooks, or evaluated against the teacher's "keys" to their tests.

They seemed to feel that a clear and simple answer was required for their posters. They did not value their mathematical reasoning and therefore did not record their fascinating explorations on their posters. Instead, they typically surrounded weak representations of their strong dialogue with decoration. The responsibility for these students' misplaced values rests with all of their past and present mathematics teachers.

In both classes, because of the students' familiarity with problems that call for one answer, and because of their familiarity with limited time that only allows shallow experiences with their problems, they were surprised by the complexity of the mathematical spaces into which they were propelled by the two investigative projects. They were as disoriented as tourists would be to live for a while in the simple-looking Swazi homesteads they are accustomed to viewing from their insulated tour buses.

Student Writing

I suggest that, because of the complexity of the mathematical territory, student writing that comes from their exploration uncovers values. Since there were numerous possible approaches to both their explorations and their explanations, the students' writing reveals their choices, which are reflections of their values. In more shallow experiences of mathematics, these values are more likely to remain covered because students are not asked to make choices about what aspects of their thinking to write up.

In this discussion of student writing, I show a sample of student writings that seem to be missing various important aspects of their authors' mathematical thinking. The first of these is the strongest example of decoration having been preferred over mathematical interests. The authors of the next one showed an interest in writing but they stumbled with inability to express their thoughts. The third poster appears to show little interest in writing, but I suggest that the reason for this void might not be obvious by simply looking at the poster.

Both teachers in this research opted to have students submit a common poster. This approach is unlike the typical British investigation practice in which students typically work cooperatively in exploration and independently in the preparation of individual write-ups. I chose to make no comment on this option exercised by Mrs. Foster and Mr. Penner, partly because I was trying to avoid interference in their decisions and partly because I hoped that if students were required to submit a common poster they would discuss their choices about what to write.

After her students worked on “Parallel Division”, Mrs. Foster and I discussed her students’ writing. She was curious about how they decided what to put on their posters, so she gave them a questionnaire the next time the class met, asking among other things, “how did you determine what materials went on your poster paper and what materials stayed on the scrap paper and went into the garbage?” We hoped we would gain insight into what her students valued in mathematical thinking.

The majority of responses to this question reflected her students’ lack of awareness of their choices in deciding what to write. This response typifies such responses:

> We basically made a rough draft of what we were going to do on scrap and edited it. Then we planned out how and where we were going to put everything. It all came together at the end.

This student gave longer answers than most other students – apparently trying to take the teacher’s questions seriously – but described layout rather than content choices.

Other students’ values were uncovered in their response to their teacher’s question, but I cannot be sure whether these students were conscious of the values reflected in their responses. These students demonstrated that they did not value their conjecturing and testing. Rather, they only valued clear and simple answers. Figure 17 lists excerpts from some of these responses.

Supporting the results of this questionnaire, the posters that students submitted also provide evidence of what these students considered to be valued mathematics. Or, considering Morgan's (1998) assertion that students write with a target audience in mind (p. 46), it might be more appropriate to conclude that the posters provide evidence of what these students expected their teachers to value.

Mrs. Foster's question

Since all of your questions on the project asked you to explain, how did you determine what materials went on your poster paper and what materials stayed on the scrap paper and went into the garbage?

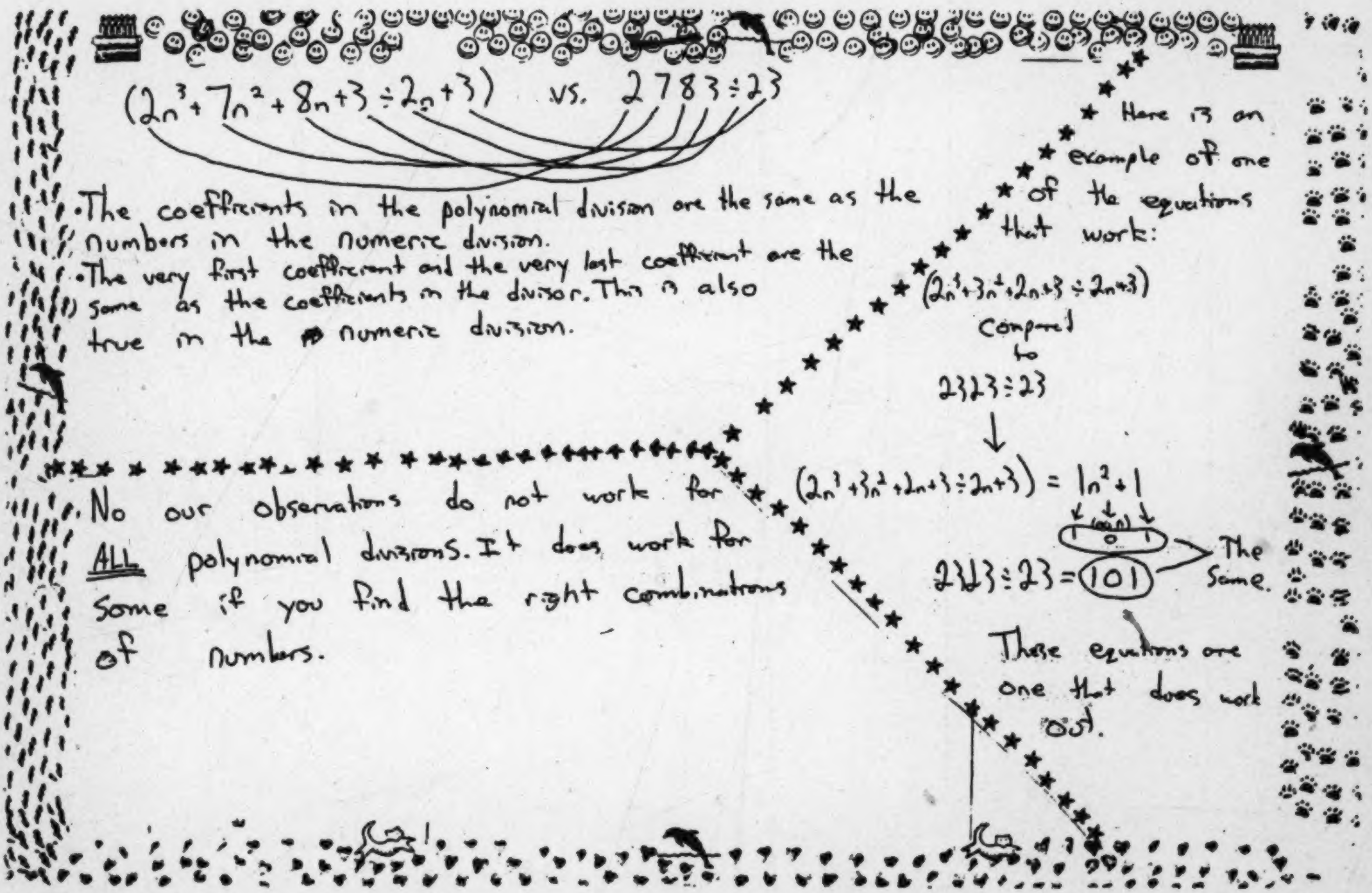
Excerpts from anonymous student responses

- We just put our findings on the paper – none of our work.
- The examples that we tried to work out went in the garbage, and not onto the poster paper. We really never thought of putting it on the final project.
- The stuff that went into the garbage were the different examples we tried and just the brainstorming. The stuff on the poster was what we decided was final.
- All our scribbly work went into the garbage. Only the stuff that was accurate went on the poster.

Figure 17. Some student reflections on valued mathematical text

Earlier in this chapter, I describe the disagreement between Aaron and Brent over their readiness to start writing. Aaron was more interested in the problem, and Brent was more interested in getting something on paper. Their submission, copied in Figure 18, is a poor reflection of the depth of the exploration of Aaron and his group. They only presented an example that fits their initial, most simplistic conjecture, rejecting examples that did not fit. They neglected to include a discovery they liked enough to call people over to see; Brent showed me that both $2783 \div 23$ and its palindrome $3872 \div 32$ have workable parallel polynomial divisions, and have a common quotient, 121.

Furthermore, instead of describing the parallels between the algorithmic work of the two kinds of division, this group merely repeated the task's description of the parallels between the two division questions. Instead of including the mathematics that fascinated them, they opted to fill their poster with decoration. Perhaps Aaron's



$(2n^3+7n^2+8n+3 \div 2n+3)$ vs. $2783 \div 23$

- The coefficients in the polynomial division are the same as the numbers in the numeric division.
- The very first coefficient and the very last coefficient are the same as the coefficients in the divisor. This is also true in the numeric division.

No our observations do not work for ALL polynomial divisions. It does work for some if you find the right combinations of numbers.

Here is an example of one of the equations that work:

$(2n^3+3n^2+2n+3 \div 2n+3)$

compared to

$2323 \div 23$

$\downarrow$

$(2n^3+3n^2+2n+3 \div 2n+3) = 1n^2+1$

$\overset{(0n)}{1 \quad 0 \quad 1}$

$2323 \div 23 = \boxed{101}$

The same.

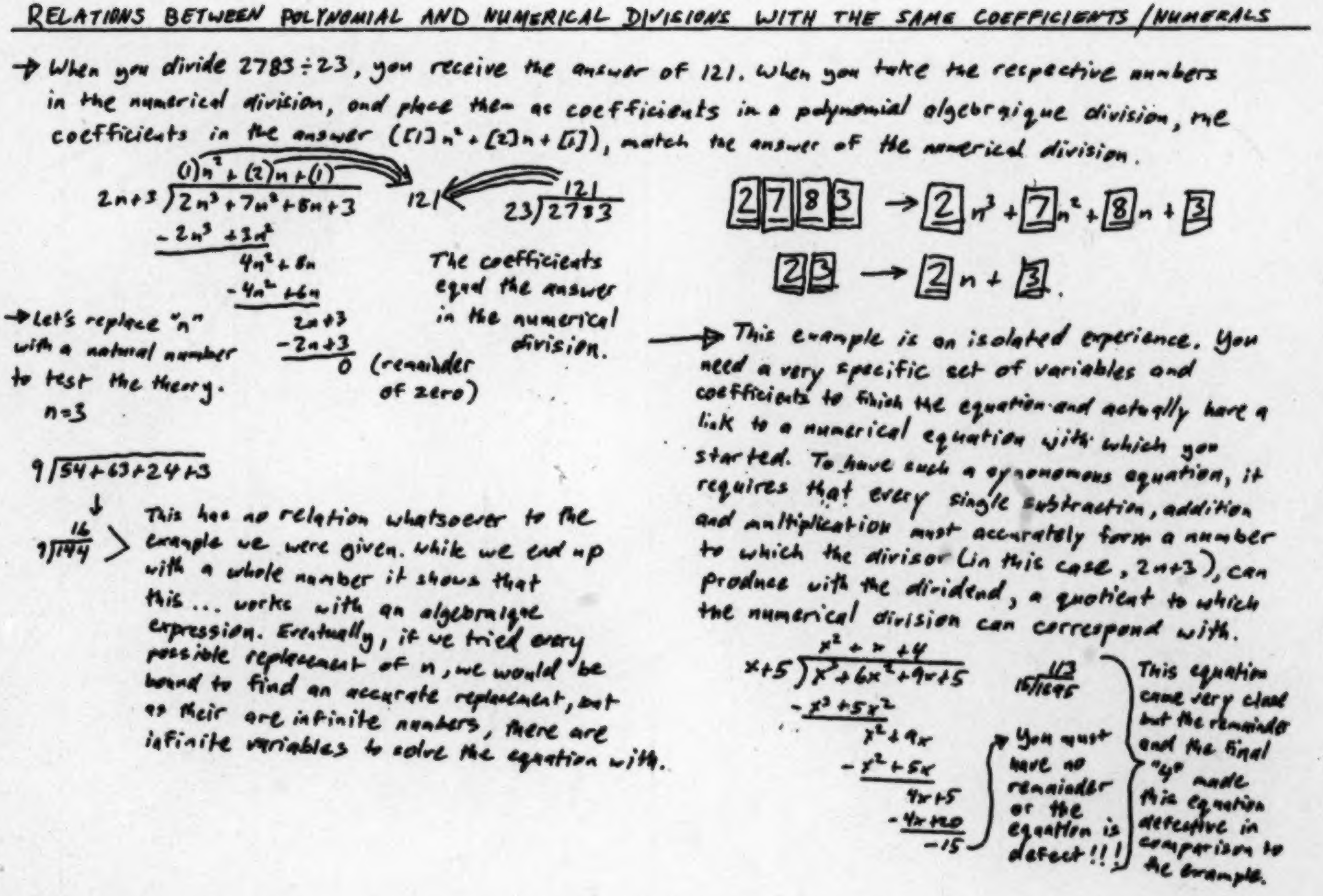
These equations are one that does work out.

Figure 18. "Parallel Division" work by Aaron, Brent and Don resistance to considering himself ready to write led to their relatively vacuous submission.

The written submission of a group of Mr. Penner's students looks markedly different. It is filled with writing. As much as the previous group's lack of writing reveals its members' values, the writing on this poster reflects the values of its authors. This group had no interaction with their teacher during their work, so they must have been basing their understanding of their teacher's expectations on their previous experiences with him or with their previous teachers. Figure 19 is a copy of their poster.

This group felt compelled to write down their thoughts, but they found the task difficult. It is likely that they had had little experience writing in mathematics. Their writing seems similar to the oral discourse that I expect to have taken place in their past mathematical experiences.

This group's readiness to write provides an opportunity to look more carefully at the language they used to represent their mathematical thinking. They used the generalization sense of the personal pronoun *you* to describe their mathematical findings. The absence of first person pronouns is most likely attributable to the acculturation they had previously experienced in school, but with the absence of their personal voice, they seem to have ignored the significance of themselves as actors in their investigation. Perhaps they assumed that anyone else would see and do the same as they had – that there is one answer, and they needed only to find it. Or, perhaps they were merely modelling the mathematical language to which they had been exposed. In this particular case, the modelling possibility is less likely than it might be elsewhere because their teacher, Mr. Penner, often used the first person singular in his regular teaching. For



RELATIONS BETWEEN POLYNOMIAL AND NUMERICAL DIVISIONS WITH THE SAME COEFFICIENTS / NUMERALS

→ When you divide $2783 \div 23$, you receive the answer of 121. When you take the respective numbers in the numerical division, and place them as coefficients in a polynomial algebraique division, the coefficients in the answer $([1]n^2 + [2]n + [1])$, match the answer of the numerical division.

$$\begin{array}{r} (1)n^2 + (2)n + (1) \\ 2n+3\,\overline{)\,2n^3+7n^2+8n+3} \\ \underline{-2n^3+3n^2} \\ 4n^2+8n \\ \underline{-4n^2+6n} \\ 2n+3 \\ \underline{-2n+3} \\ 0 \end{array} \quad 121 \quad \begin{array}{r} 121 \\ 23\,\overline{)\,2783} \end{array}$$

(remainder of zero)

The coefficients equal the answer in the numerical division.

$$\boxed{2}\boxed{7}\boxed{8}\boxed{3} \rightarrow \boxed{2}n^3 + \boxed{7}n^2 + \boxed{8}n + \boxed{3}$$

$$\boxed{2}\boxed{3} \rightarrow \boxed{2}n + \boxed{3}.$$

→ Let's replace "n" with a natural number to test the theory.

$n=3$

$$9\,\overline{)\,54+63+24+3}$$

$$\downarrow$$

$$\begin{array}{r} 16 \\ 9\,\overline{)\,144} \end{array}$$

This has no relation whatsoever to the example we were given. While we end up with a whole number it shows that this... works with an algebraique expression. Eventually, if we tried every possible replacement of n, we would be bound to find an accurate replacement, but as their are infinite numbers, there are infinite variables to solve the equation with.

→ This example is an isolated experience. You need a very specific set of variables and coefficients to finish the equation and actually have a link to a numerical equation with which you started. To have such a synonomous equation, it requires that every single subtraction, addition and multiplication must accurately form a number to which the divisor (in this case, $2n+3$), can produce with the dividend, a quotient to which the numerical division can correspond with.

$$\begin{array}{r} x^2 + x + 4 \\ x+5\,\overline{)\,x^3+6x^2+9x+5} \\ \underline{-x^3+5x^2} \\ x^2+9x \\ \underline{-x^2+5x} \\ 4x+5 \\ \underline{-4x+20} \\ -15 \end{array} \quad \begin{array}{r} 113 \\ 15\,\overline{)\,1695} \end{array}$$

You must have no remainder or the equation is defect!!!

This equation came very close but the remainder and the final "4" made this equation defective in comparison to the example.

Figure 19. "Parallel Division" work by Glen, Wes, James and Jason example he asks, "What would I do next in this situation?" However, his use of the pronoun *I* seems to resemble the more typical use of *you* for generalization.

Another value displayed in this group's poster seems to be characteristic to both classes' work on "Parallel Division". Students seem to have been more interested in making a statement than in testing and showing their statements' validity. Aaron was an exception to this tendency with his refusal to write anything at all.

Jason's group's statement, "You must have no remainder" is not fully true. It cannot be generalized. I assume that they believed it to be true, but it seems likely, from my own exploration, that if they had tried even a limited number of examples they would have found instances where the parallel holds with a remainder. For example, if I were to add "1" to the task's given example, the parallel would hold with the quotients remaining the same and with the same remainder in both the polynomial and whole-number division. $(2n^3 + 7n^2 + 8n + 4) \div (2n + 3)$ is equal to $n^2 + 2n + 1$ with a remainder of one, and $2784 \div 23$ is equal to 121, also with a remainder of one.

In Jason's group, as in many others, the members worked independently with only occasional conversation about their findings. It is likely that they were looking at each others' work more than they were talking to each other. Because of their relative silence, it is impossible for me to discern from the audio record the extent to which they tested their conjecture about remainders.

More than demonstrating these students' values, another feature of this group's writing might demonstrate the values promoted by their current and former mathematics teachers. These four boys seemed to struggle to find words to describe the ideas that they visualized. When preparing to write an idea on their poster, Jason exploded with a rant in which he described his frustrations: "How the heck do you present your findings? It's in your mind! Here, read this. [*He laughs sarcastically*]."

It seems that too many of their previous experiences of mathematical places had not required them to explain their thinking. The difficulty they experienced is most evident in this baffling excerpt from their poster:

> To have such a synonomous [*sic*] equation, it requires that every single subtraction, addition and multiplication must accurately form a number to which the divisor (in this case, $2n + 3$), can produce with the dividend, a quotient to which the numerical division can correspond with.

These boys have not had enough experience with verbal pointing. I expect that their statement would be greatly supported with their pointing fingers drawing a listener's attention to the appropriate parts of an example.

At first read, I thought I understood what these boys were trying to explain, but now I am reminded of Morgan's (1998) revelation that teachers construct explanatory narratives for mysterious student writings (p. 182). I am reading my understanding of parallel divisions into the writing.

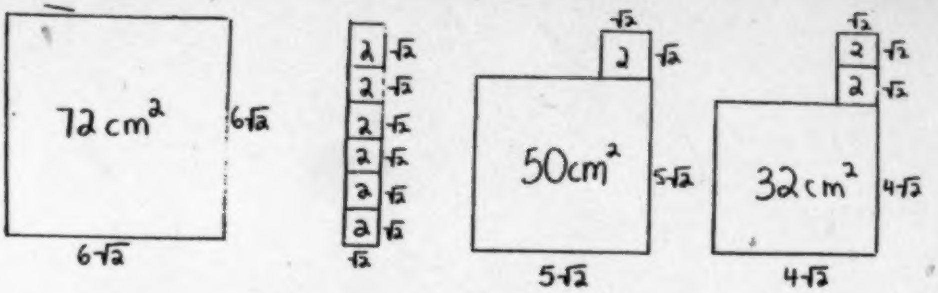
This group differs from many of the others in that they gave examples of their statements. They provided examples to support their claims that for parallel divisions to work there can be no remainder, and that most replacement values for n have no significance. Both these claims are not justifiable generalizations, but in the spirit of time-limited exploration I believe that it is unfair to expect perfectly constructed conjectures.

Other groups provided examples with scant explanation. For instance, in their "Playing with Squares" work, Shawn, Terry and Brian in Mr. Penner's class found more examples of stacks of squares than any other group in either class, but provided almost no explanation. They found themselves so engaged in the first instruction in their task – finding all the stacks that matched the 72-square in height – that they had little time left for the remaining instructions, which called for more writing. For an observer who merely looks at their poster, their scant writing might be seen as an indication that they did not value the verbal aspects of mathematics. However, the audio record indicates that their lack of writing was more likely due to their careful response to the first prompt. This group is the only one that seemed to feel inspired to find *all* the matching stacks. Other groups were more interested in other areas of exploration and some, in their oral discourse, explicitly rejected the idea of finding all the matching stacks for the 72-square. This excerpt from the dialogue of Shawn, Terry and Brian demonstrates their focus on exhausting the possibilities:

> Shawn: Now all we're missing is four root two plus two root two [$4\sqrt{2}+2\sqrt{2}$] ...if we're doing it right.
> Terry: We are.

Their poster is shown in Figure 20. Although this group of boys was consumed with the task of finding all the matching stacks, their poster shows only three. The audio tape of their work indicates that they found others. The poster resembles the one made by Natalie, Kathy and Teresa, who provided a couple of examples but no words whatsoever (see Figure 10 on page 68). Although the posters are similar in their paucity of writing, each case has its own reasons.

By looking at the posters in these two investigative environments, it seems that the students had varying interpretations of what aspects of their mathematical thinking were most important to share. In her study, Morgan (1998) found a phenomenon different from these students' apparent inability to read their teachers' expectations. In the



- No, because you couldn't make another square you could only make rectangle
- By putting different square roots together to to equal $6\sqrt{2}$.

Figure 20. "Playing with Squares" work by Shawn, Terry and Brian

formalized British investigation environment, teachers had a very clear idea about what they valued in mathematical write-ups, but these expectations were not necessarily made explicit for students. Students became aware of these expectations through their teachers' modelling and grading.

It is unclear to me from Morgan's (1998) work to what extent student write-ups matched their teachers' expectations. I assume, from her assertion that students write with their teacher-as-audience in mind, that, in her experience, student mathematical texts typically corresponded, to some extent, with typical teacher expectations.

Although the participants in my study differed from typical British students in the extent of their experiences with writing in mathematics, they all seem to share a relative lack of fluency. Morgan (1998) reports that teachers in the United Kingdom agree on this point: "pupils find it difficult to communicate the full extent of their investigative activity in writing" (p. 72). Mr. Penner expressed similar feelings about his students' weak written representations when he decided to look at their scrap paper along with their posters in the first project. Similarly, Mrs. Foster, when she administered a questionnaire to her students, searched for clues to her students' deficient writing.

Morgan (1998) concludes her research into mathematical writing with a call for clear classroom dialogue about what is valued in mathematical writing (p. 209). I suggest that her proposal would also have served well in the two classes participating in this research. I expect that if Mr. Penner and Mrs. Foster had initiated such dialogue with their students before their work on these tasks, these students' writings would have been more representative of their mathematical thinking, perhaps including their conjecturing and testing, as well as their conclusions. I am not at all surprised that they did not do this. Morgan's advice implies that such an approach is uncommon even in the United Kingdom despite its proliferation of investigative project-work. Both teachers in this research read aloud for their students the expectations from the scoring rubric's standard of excellence, but neither provided examples from which they could have discussed valuable features.

The students in these classes appeared to be unfamiliar with open mathematical landscapes in which they could look in any direction. Various students were captivated by different possibilities within the landscape, and students had different ideas about which features in the territory demanded their attention. Because the complexity of the space allowed them to direct their attention in different ways, the directions they chose reveal their values to some extent. What is important to notice? What is important to report?

Creativity

Because a complex landscape cannot yield simple answers, participants are compelled to exercise creativity. In Chapter 4, I suggest that condoms are a simple answer to the problem of AIDS – the kind of simplistic answer characteristically heard from people unconnected with the problem-connected place. Participants would have so many questions about the "condoms" answer that they would not know where to begin asking for clarification from an ignorant, unconnected tourist. With the rejection of simplistic answers, participants in the Swazi community open for themselves a space for creative exploration. As soon as we register a simple answer as *the* answer, we close ourselves off from considering other possibilities.

Although the participants in the two investigative mathematics projects in my study seemed to feel bound, not freed, by the complexities and associated ambiguities of the open mathematical landscape laid before them, the restrictions appeared to let the students loose to find some of their creativity. Indeed, it is a participant's connection – the reality of being bound to the problem – that motivates the rejection of simplistic answers, and inspires the resultant search for creative responses based on heightened awareness.

The creativity of Natalie and her group became available to them when they decided to resist the simple approach of asking for help. Other students also entered such zones of creativity in their responses to the investigative tasks. In this discussion of creativity, I focus first on student responses from the "Playing with Squares" environment. I then touch on the "Parallel Division" experience. It seems that the

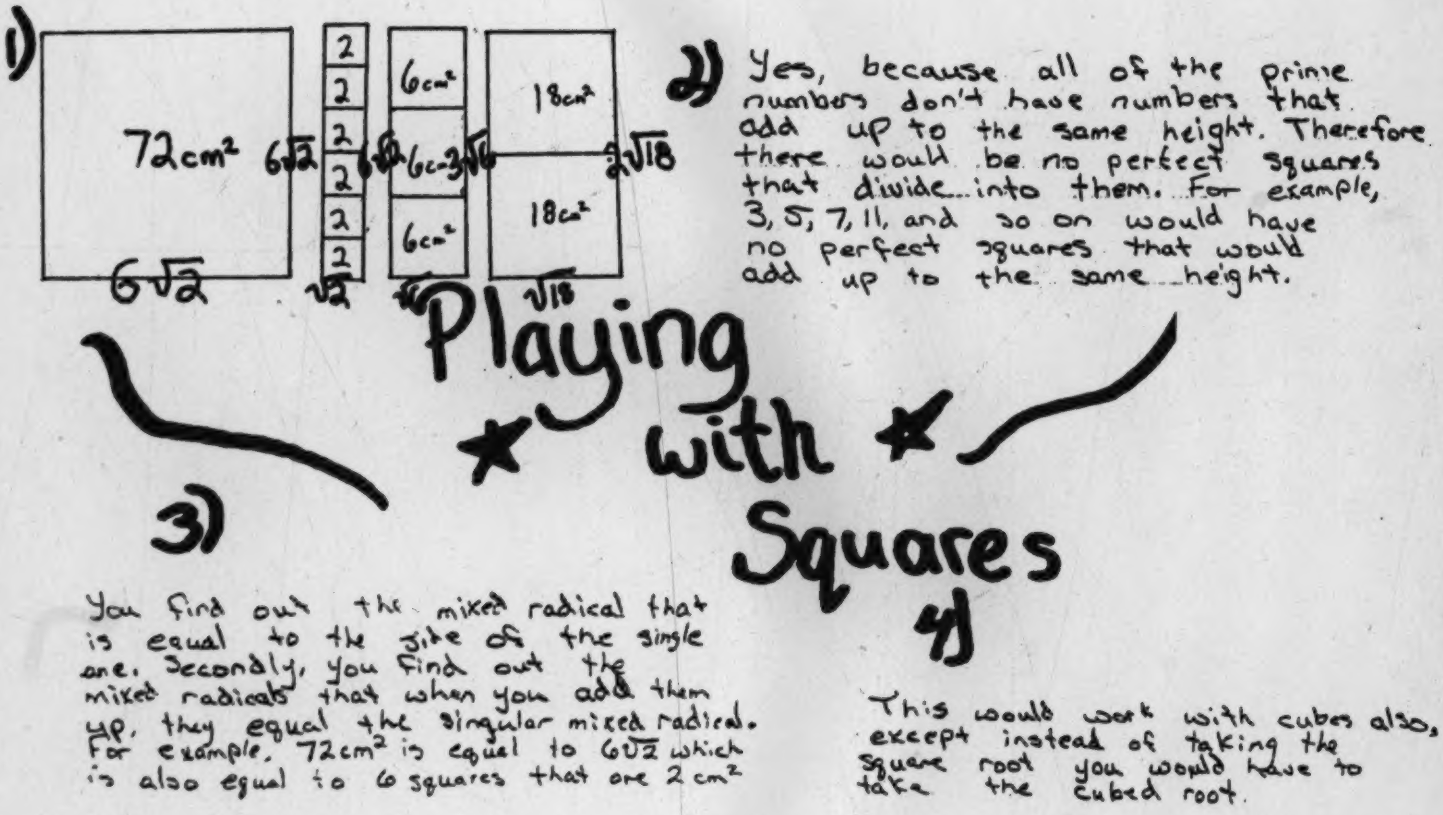
"Playing with Squares" task was the more successful of the two tasks in terms of drawing out student creativity.

Before presenting some of the more creative student findings, I look at what I had expected to be a typical response to "Playing with Squares". It seems that both teachers were also expecting this kind of approach. Judging by the various approaches in student posters, it becomes clear that these students were seeing the mathematical landscape opened by "Playing with Squares" quite differently from the way we mathematics educators had expected.

The students who submitted the poster copied in Figure 21 considered the heights of the squares in terms of their mixed radical representations. These students seemed to have recalled their earlier classroom work in which they had simplified pure radicals and added mixed and pure radicals. With a readily-apparent solution in mind, these students were not consumed by the problem, not driven to exercise their creativity, and not careful in their explanation. For a problem to be problematic – for it to be a real problem – it cannot have an obvious simple solution.

The group that wrote this text, like Jason's group, implied generality by using the pronoun *you* in its generalizing sense. They seem to have felt that the solution was obvious enough that one set of examples would suffice – the set of same-sized square stacks that match the 72-square – and that no clear demonstration of their symbolic manipulation was necessary.

With a look at other responses to the same task, it is clear that although both teachers expected this approach, the approach was not so obvious to other students. Natalie's group, for example, was not only propelled into a zone of creativity by its


72cm²
6√2
6√2
2
2
2
2
2
2
√2
6√2
6cm²
6cm²
6cm²
3√6
√6
18cm²
18cm²
2√18
√18

Playing with Squares

1)

2) Yes, because all of the prime numbers don't have numbers that add up to the same height. Therefore there would be no perfect squares that divide into them. For example, 3, 5, 7, 11, and so on would have no perfect squares that would add up to the same height.

3) You find out the mixed radical that is equal to the size of the single one. Secondly, you find out the mixed radicals that when you add them up, they equal the singular mixed radical. For example, $72 cm^2$ is equal to $6\sqrt{2}$ which is also equal to 6 squares that are $2\ cm^2$

4) This would work with cubes also, except instead of taking the square root you would have to take the cubed root.

Figure 21. An expected response to "Playing with Squares"

decision to avoid asking for help, but also because the "obvious" approach displayed in Figure 21 was not available to them. The upcoming samples demonstrate that other students also played with squares in zones of creativity in both classes. These students had no help from their teachers. In relation to these students, their teachers were able to maintain their positions as outfitter guides.

Gordon revealed an approach to the "Playing with Squares" problem that has continued to fascinate me long after his experience. I watched Gordon play with squares for a while. He was completely silent, so I had no access to his thoughts except through my reconstruction of what I saw him do. He was stumped by the problem, so he decided to cut squares out of coloured paper to paste onto his group's poster. First he made a couple of large squares, which he duly labelled "72" to represent their areas, seventy-two square centimetres. He then realized that he only needed one of these, so he decided to make smaller squares out of one of these larger squares. He cut it vertically and horizontally and labelled each of these smaller squares "18" because they were a quarter the size of the 72-square.

He suddenly became entranced by his own manipulations. He realized that he had made two stacks of 18-squares that were the same height as the 72-square – he had just found a viable answer to one of the questions in the task using a very concrete method. He moved one of these stacks of 18-squares beside the 72-square and moved it back to its source position. In awe, he repeated this movement again and again, about five times. Figure 22 is a static attempt at diagramming Gordon's dynamic constructions.

After reflecting for a while on Gordon's approach to making matching stacks, I have come to realize that it has enormous potential for providing students with hands-on

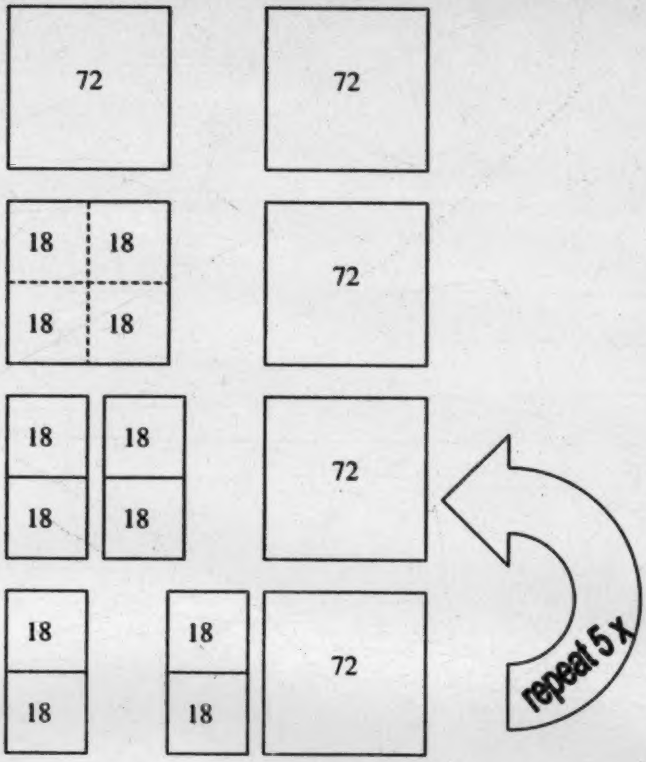


Figure 22. Gordon's stacks of squares

experiences that can help structure their understandings of radical arithmetic. Since watching Gordon perform this seemingly magical construction, I have led two teacher groups through his processes. These teachers seemed to be as impressed as I with the potential in the process.

The 72-square can be divided into four, nine or thirty-six equal-sized squares. These are the perfect square factors of seventy-two ($2^2 \times 18 = 72$, $3^2 \times 8 = 72$ and $6^2 \times 2 = 72$). After cutting out stacks like Gordon did, students could have a different experience with simplifying pure radicals: $\sqrt{72} = \sqrt{4} \times \sqrt{18} = 2\sqrt{18}$, $\sqrt{72} = \sqrt{9} \times \sqrt{8} = 3\sqrt{8}$ and $\sqrt{72} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}$.

With different arrangements of the cut-out squares, we can model radical arithmetic. Figure 23 models the expression $\sqrt{72} = \sqrt{8} + 4\sqrt{2}$. Gordon did not push his idea this far. However, I have the benefit of months of consideration after my amazement with his seemingly simple technique – an advantage that he did not have in his 70 minute investigation.

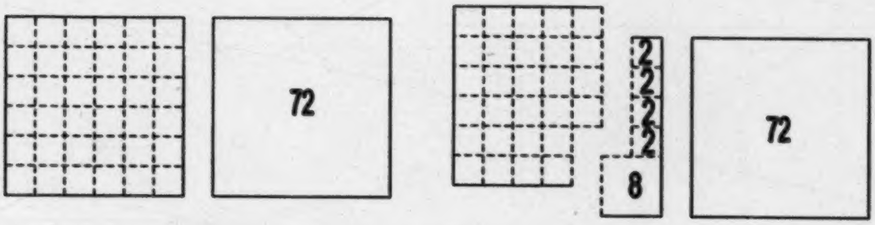
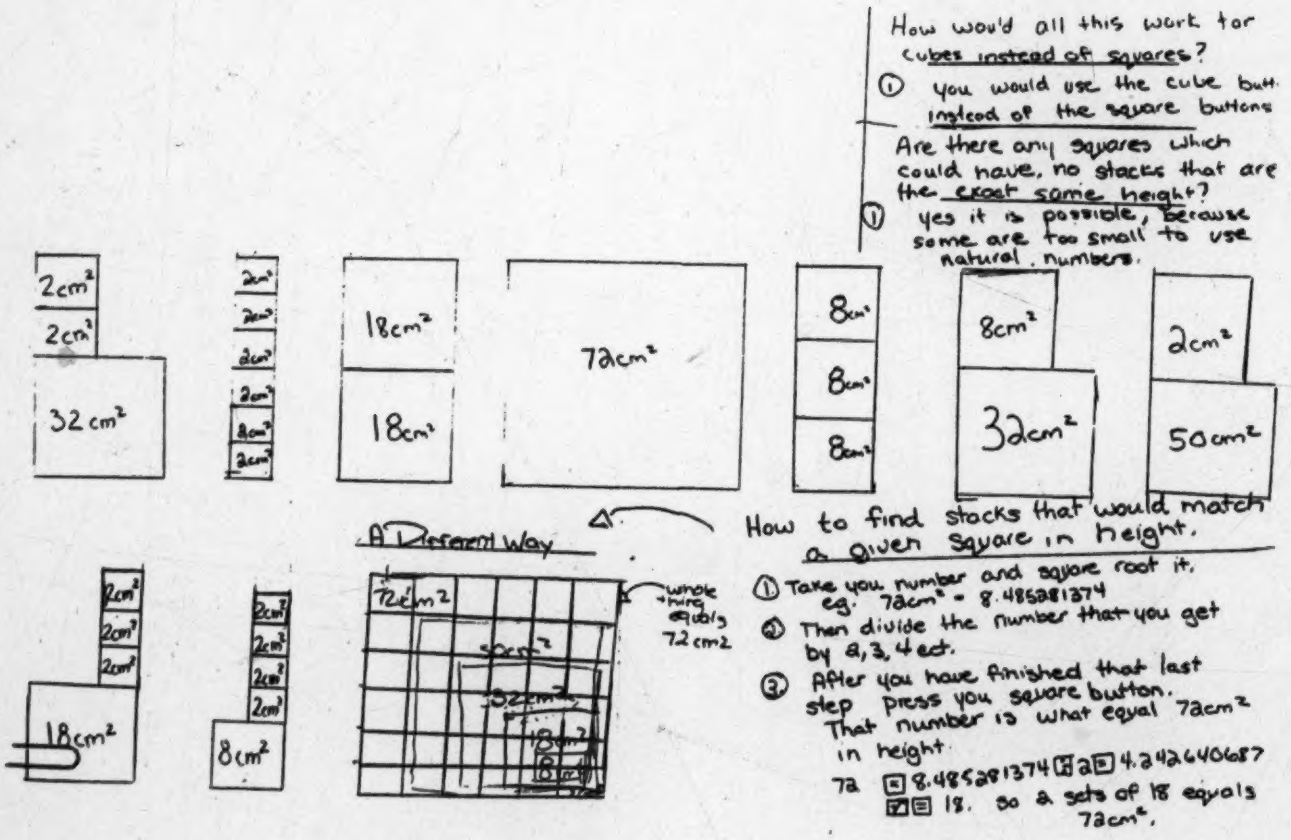


Figure 23. Modeling $\sqrt{72} = \sqrt{8} + 4\sqrt{2}$

The next sample of student writing, copied in Figure 24, demonstrates two different approaches to the same problem, neither of which depend on the addition of radical numbers. This group received no help from their teacher, Mr. Penner, only his two-word affirmation near the end of the project time: "Good work".

The "How to find stacks…" method this group presents is described verbally with a clear connection between their method and the example they used to demonstrate the method. The writer of this set of instructions must have been conscious of the necessity of making these connections explicit. Perhaps Chantelle's experience of trying to explain her idea to her groupmates helped them to realize the need for an example. In the first half of this explanation she tried to describe her idea to her friends without an example. When she realized that they could not find meaning in her description, she repeated her


2cm²
2cm²
32 cm²
2cm²
2cm²
2cm²
2cm²
2cm²
2cm²
18cm²
18cm²
72cm²
8cm²
8cm²
8cm²
8cm²
32cm²
2cm²
50cm²
A Different Way
72cm2
50cm²
32cm²
18cm²
8cm²
whole thing equals 72cm2
2cm²
2cm²
2cm²
18cm²
2cm²
2cm²
2cm²
2cm²
8cm²

How would all this work for cubes instead of squares?

① you would use the cube butt. instead of the square buttons

Are there any squares which could have, no stacks that are the exact same height?

① yes it is possible, because some are too small to use natural. numbers.

How to find stacks that would match a given square in height.

① Take you number and square root it.
eg. 72cm² = 8.485281374

② Then divide the number that you get by 2, 3, 4 ect.

③ After you have finished that last step press you square button. That number is what equal 72cm² in height.

72 ☐ 8.485281374 ☐ 2 ☐ 4.242640687 ☐☐ 18. so 2 sets of 18 equals 72cm².

Figure 24. "Playing with Squares" work by Jennifer, Kara, Tasha and Chantelle explanation, pointing to an example. Before this idea they had found two matching stacks by trying random numbers. She uses one of them for her example.

> There's an easier way to do this. You just go like, 72 square [rooted], right? You just take a number, right, divided by 9 or whatever, that would give you the number you want, and then you go squared. And if it gives you a number then that's the number. Do you know what I mean?
>
> [*after a period of quiet*] No? Okay, you know how I did two, right? So like 72-squared [rooted], like, that, no square-root, right? And then I divided by 6 to give me that and then I just pressed "squared" and if it comes out to a natural number then that's what the number is. The number I divided by gives you the number of squares.

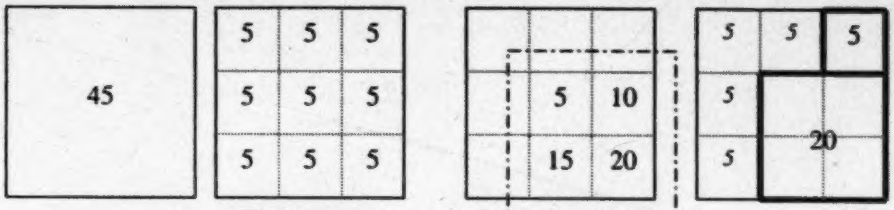
This group, like the authors of the work in Figures 19 and 20, imply generality by using the generalizing sense of the pronoun *you*.

Their second approach resembles Gordon's. They chose to represent this approach with no verbal description, only diagrams. I think that the diagrams in their poster are intended to demonstrate how they could divide up the initial square into smaller squares and subsequently find different kinds of stacks. I am unsure to what extent my access to the audio record facilitates my interpretation of their diagrams.

Tasha discovered the process that I see in their diagrams, and demonstrated it to her friends using the 45-square from the example in the investigation prompt.

> There's one like 45 right? You know how there's five, five, five or whatever? Put the fives in here, right? Like that, right? You see the fives? Like, these are all individual fives, right? So, 5, 10, 15, 20, 25, 30, 35, 40, 45. So all fives, but with this one you can do that, right? Put 5, 10, 15, 20 in. Just do that with 72.

Figure 25 is a representation of my imagined picture of the images to which Tasha seemed to be referring.



45 divided into nine equal parts

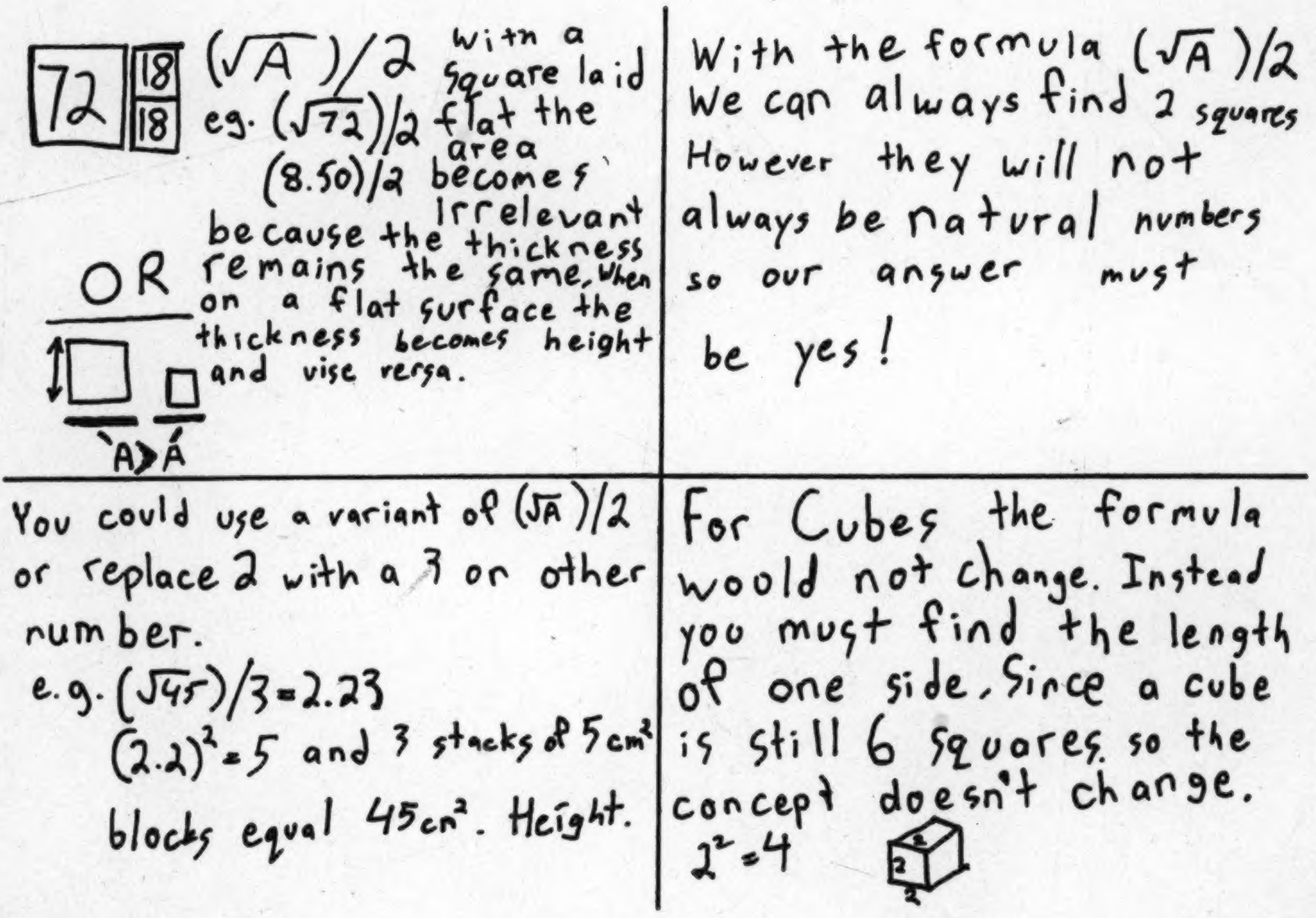
Find the sum counting by 5's i.e. 5, 10, 15, 20, 25, 30, 35, 40, 45

Find the sum of these four squares i.e. 5, 10, 15, 20

Figure 25. Figures at which Tasha might have been pointing

Jon's group enjoyed reporting about a very different interpretation of the instructions, and also reported an algebraic representation of the method written out by Chantelle's group. Jon looked at the squares in "Playing with Squares" from another perspective, literally. In response to the instruction, "Find stacks of squares that would be the exact same height as a square with area 72 cm^2", he noticed that the squares would have no height whatsoever if they were laid flat instead of upright, because they are two-dimensional. He could pile as many laid-flat squares as he would want on top of a 72-square and the pile would be the same height as another 72-square. He described his idea in this way: "With a square laid flat the area becomes irrelevant because the thickness remains the same". Figure 26 is a copy of the work submitted by Jon's group.

Their algebraic representation is especially interesting to me because of Morgan's (1998) research which reveals that teachers in Britain privileged algebraic representations of generalizations (p. 58). Jon's group was the only one in either class to attempt an algebraic representation of their generalization. Although their formula is incomplete – they should have squared their expression, $\left[\left(\sqrt{A}\right)/2\right]^2$ – their teacher, Mrs. Foster, awarded them the highest mark in the class. With their limited diagrams and



72

18

18

$(\sqrt{A})/2$ with a square laid flat the area becomes irrelevant

eg. $(\sqrt{72})/2$

$(8.50)/2$

because the thickness remains the same, When on a flat surface the thickness becomes height and vise versa.

OR

$\grave{A} > \acute{A}$

With the formula $(\sqrt{A})/2$ We can always find 2 squares However they will not always be natural numbers so our answer must be yes!

You could use a variant of $(\sqrt{A})/2$ or replace 2 with a 3 or other number.

e.g. $(\sqrt{45})/3 = 2.23$

$(2.2)^2 = 5$ and 3 stacks of 5 cm^2 blocks equal 45 cn^2. Height.

For Cubes the formula woold not change. Instead you must find the length of one side. Since a cube is still 6 squares. so the concept doesn't change.

$2^2 = 4$

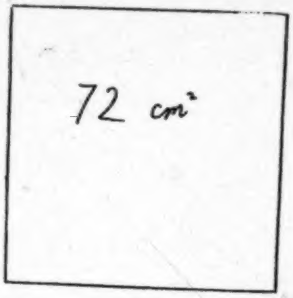
Figure 26. "Playing with Squares" work by Jon, Aaron and Brent nearly-correct formula, their teacher valued their work the highest. Arguably, the teachers Morgan studied would have evaluated these written submissions in much the same way.

The next sample of student work demonstrates another different approach and also points to another important consideration in investigative projects. It comes from a group in Mr. Penner's class. All four of its members were designated English-as-a-second-language (ESL) students. One of them was more fluent in English than the others, so it seems likely that she did the writing. The others could barely communicate in English. This group's audiotape is beyond my comprehension as it contains a smattering of four different languages. These students' mother tongues were Korean, Mandarin and Cantonese. They relied on English and Cantonese to facilitate dialogue between members with different language backgrounds. I assume that students who do not share facility in a common language with their peers would have more difficulty with investigative tasks that are intended to draw out communication. Although their poster does not seem out of place among those produced by other groups, their difficulty with the English language probably added challenge to the write up of their report.

This group's approach to the "Playing with Squares" task was based on ratios, but it was different from Natalie's ratio-based approach. The importance of perfect squares is more evident in their approach, but I am unsure whether they noticed the perfect squares in their scale factors. Figure 27 is a copy of their submission.

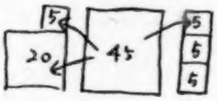
The diverse approaches to the "Playing with Squares" investigation in both classes attest to the fertility of the landscape it opens for students. Creativity in the

※The area of any square should be a Natural number



the length of each side would be : $\sqrt{72}$

We can found out the rules from the diagram above

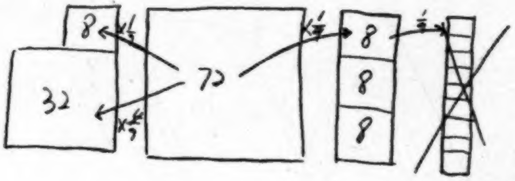


The relationships between areas (In other words. ratio between those)

$45 \times \left(\frac{1}{9}\right) = 5$

$45 \times \left(\frac{4}{7}\right) = 20$

① S.



we can't keep doing this 'cause now we can't get areas of natural numbers

② Are there any squares which could have no stacks that are the exact same height? ⇒ To find out the squares which cannot made by products of 9. For example, If the area of the square was 10, ⇒ the areas of the squares which are same height with it would be not natural numbers

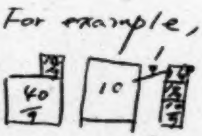


Figure 27. "Playing with Squares" work by four ESL students

"Parallel Division" task expressed itself differently. In this task, students were compelled to become creative in their conjecturing, and also in their approaches to adjusting elements of the example in the instructions.

I state earlier that many groups felt that whole-number divisions that resulted in a remainder would not have workable parallel polynomial divisions. Although some of these groups provided examples of polynomial divisions that seemed to verify this conjecture, they were not able explain why their idea might be true. Earlier in this chapter, I show, with one workable counter-example, that the conjecture cannot be justified generally.

Figure 28 lists some other conjectures students made in response to the "Parallel Division" investigative project. In the column on the left, I present these conjectures as the students have written them, and in the column on the right I comment on the conjectures.

It is difficult for me to comment on the extent to which students tested their generalizations because most groups' audiotapes are characterized by extended periods of silence, interspersed with brief comments like "Oh", "Hmm", and "Here's one". Considering the audio record, I can say that students in many groups were working independently on paper and looking at each other's work. If I judge by the lack of discussion or accuracy of the conjectures, it seems that groups did not test their conjectures carefully. They seemed content to merely arrive at and articulate their ideas.

One group explicitly discussed conjectures – Jennifer, Kara, Tasha and Chantelle in Mr. Penner's class. Shortly after they found that not all whole-number division

Student conjectures as reported on their write-ups of their "Parallel Division" work	My comments
"If all the numbers and variables are positive then it will work and if they are all negative it will also work but you must change the numbers in the numeric division to negatives for it to work."	This is basically true, as I understand it. In effect, these students said that either all the coefficients ought to be positive, or they all ought to be negative. It would be even more accurate to say that such cases *can* work and hence avoid saying that these cases would always work.
"We found that replacing the signs didn't affect the equation in the slightest, so long as the format stayed the same."	This is not true. The example that this group provided to support their conjecture seems to be a case where it is true, but the calculation in the example is fraught with arithmetic errors.
"We found that polynomials with 4 terms basically a cubed as highest value worked, but any more or less didn't."	This is not true. The given examples support the conjecture, but these are special cases.
"You must have no remainder or the equation is deficit!!!"	About half of the groups made statements that seem similar to this one. This particular statement might not be an attempt at a generalization, but this group's use of the pronoun *you* elsewhere in their presentation seems to be in the generalizing sense. From the audio record, it seems that this group derived this generalization from one example, and that they did not test it with other examples.
"All polynomials are not the same and some end up with remainders or just don't work out. The few ones that do work out have to have a 2 at the end of each of the polynomials."	The generalization is not true, and the group does not even provide a single example that seems to substantiate it.
"the ones it does work for has to have the 1st and last number in the dividend to be one number apart from each other and the two middle numbers have to be only 1 number apart from each other. Then the divisor is the first and last number of the dividend."	This set of statements cannot be generalized. I suggest that this group would have benefited from asking why their supporting examples had these characteristics. The last sentence has potential connections with the Remainder Theorem.

Figure 28 – Student conjectures regarding "Parallel Division"

statements had parallel polynomial division statements, they noticed that divisions that had perfect-square quotients had parallels and divisions that did not have perfect-square quotients did not. They tested this idea by working backwards to construct divisions that would have perfect-square quotients. Their first examples supported their hypothesis, but the group persevered with enough testing to realize that the hypothesis did not hold. Their thirteen minutes of work on this hypothesis was rich with mathematical activity, especially in their backwards design of division questions that would result in perfect-square quotients. I consider it unfortunate that they did not value this work enough to report on the possibility they saw and their methods of testing.

Perhaps if the mathematical culture in these two classes had valued such work, then the other groups might have benefited as well. If other students had valued conjecturing and testing enough, perhaps they would have been motivated to test their conjectures more thoroughly.

Teachers might expose their students to models of mathematical thinking by looking together at exemplars of investigation reports. In such a context, they could discuss valued features of student writing and engage in dialogue about what makes mathematical writing valuable. I believe that this kind of discussion is what Morgan (1998) has in mind in her call for critical language awareness (p. 209). Teachers might also model conjecturing and testing by allowing themselves to be captivated by rich mathematical problems. If they were to report on their work to their students, they would be able to draw attention to the mathematical possibilities that fall out of their "mistakes".

Having been somewhat outfitted for exploration in rich mathematical landscapes, the students' unexercised mathematical creativity readily sprang to life, but their unfamiliarity with the complexity of the landscapes rendered them ill-equipped to pursue their ideas in depth. These students' readiness for creativity is especially noteworthy because their teachers did not typically encourage creativity in normal classes.

For the second project, Mr. Penner used a scoring rubric that does not consider creativity in its evaluation (see Appendix 1). Although he has convinced me that this switch was not overtly intentional, when I pointed out the difference between the two rubrics, he admitted discomfort with the idea of creativity in mathematics.

Mrs. Foster seemed slightly more comfortable with creativity during her students' project work, but the language used by her and her students also reveals one-right-way thinking. Many of her students asked her "Are we on the right track?" during their project work. A "right track" implies one right destination. Although she mentioned the possibility of shortcuts in regular classes, she did not pursue them. In one of the classes before they engaged in these projects, for example, when asked if there was another way to arrive at a particular answer – a short-cut – she told the students that they would have to figure out such approaches themselves, and then continued with her planned examples: "you gotta figure it out [yourself]", she said. For any particular problem type she would stick to one path in her presentation to students.

Summary

The richness of immersed experience carries with it an inextricable connection to the problems in the space. The problems of the landscape are characterized by complexity that is only evident to participants in the place. Because of their understanding of this complexity and their rejection of simplistic answers, the captivated participants can be propelled into a zone of creativity. Their values are exposed in this space of possibility and decision making.

The early stages of my immersion in Swaziland were marked by disorientation and frustration. Even in these times I enjoyed tremendous growth as I was released to see from different perspectives and to recognize cultural particularities. Once I became accustomed to living within my new complex community, I benefited from such growth without the frustration that accompanied the earlier stages of my experience.

The students and teachers immersed in the mathematical spaces opened up by the "Playing with Squares" and "Parallel Division" tasks experienced a disorientation that to some extent paralleled my early experiences in Swaziland. I submit that more exposure to such places would be a healthy release for them. They would grow in their new understanding of the complexities inherent to mathematical spaces and be more comfortable living amidst such ambiguity.

Chapter 7 – Reflections

In this final chapter, I summarize the other chapters and briefly reflect on directions for further exploration inspired by this research. In some ways, my investigation did not meet my initial expectations, but as I became engaged in the classroom environments that I was studying I found myself captivated by other concerns that, in the end, seemed more important than my expectations. As I began to see mathematics environments as complex, multi-dimensional spaces, I found a connection between some of my immersions in foreign places and mathematics classroom culture. I consider here what this research has done, what it did not do and what it begs doing.

It is my hope that as mathematics teachers find resonance in the stories and interpretations that I share from my research experiences, they will feel more freedom and courage to reconsider the way they view their pedagogic relationships, and perhaps modify their classroom practices to include more experiences of mathematical investigation for their students. As their students become more aware of the complexities of the problems in their mathematical places, perhaps they will become more aware of the complexity of the problems in their communities, their country and their world. It is my hope that as they become more accustomed to exercising their creativity within complex mathematical landscapes, these people will be outfitted to respond with creativity and sensitivity to the very real problems they will experience outside the mathematics classroom.

Summary

In Chapter 1, I describe the conception of my research idea. My interest in pure mathematics investigations emerged from my teaching and intercultural experiences. As I provide a backdrop for the rest of the thesis, I share my discomfort with authoritative descriptions of past events, but embrace the traditional thesis format because of its evocation of a line of progeny to which this study belongs.

In Chapter 2, I review literature that relates to one ancestor to this study, the investigation. I locate the literature in my review according to its relationships with mathematical investigations in the United Kingdom. I consider literature that looks at this phenomenon from within, and literature that uses the phenomenon to look outside. I close with a look at some North American cousins to the British experience.

In Chapter 3, I describe how I planted my research seed. I describe how I cultivated its growth knowing that I could not predict in advance exactly what would grow. I describe my readiness to welcome empty answers as a source of unexpected fullness. An empty pot is full if I am interested in truth or surprises.

Chapter 4 presents the new way of seeing that allowed me to see the empty pot as full. Overwhelmingly, the students and teachers, in their investigative project work, seemed to be disoriented – they seemed to experience the kind of disorientation that I recognize as characteristic to experiences of foreign places. My emerging perception of mathematical experience as a complex place forms the basis for my interpretation of the research participants' brief immersion in a slightly foreign landscape. I consider the implications for mathematics students guided in various ways by teachers. Teachers who protect their students from a seemingly harsh mathematical landscape provide them with shallow experience. By contrast, students who are drawn into participation in the complexity of the landscape can experience the depth and interconnectedness of mathematical problems. Teachers might accompany their students as neighbourly guides or propel them in as outfitter guides.

In Chapter 5, I present two scenes that exemplify the compelling complexity of mathematical immersion. In the first scene I describe how Mr. Penner experimented with different ways of guiding in his interaction with one group and, as a result, distracted the students. They moved out of the zone of creativity they had found in the open mathematical space he initially set up for them. This scene demonstrates how difficult it can be for teachers to choose their roles and interventions in their students' investigative work. In the second scene, Greg, who had been sheltered from the complexity of the mathematical landscape, is shown to have been mired in frustration as he engaged in high-level mathematical thinking. This second scene might direct teachers who are interested in using investigative projects to consider the frustrations that are likely to result from their students' sudden exposure to the complexity that underlies all their mathematics – a complexity they are seldom permitted to encounter.

Chapter 6 draws upon these scenes and others to colour a picture of mathematical immersion. Deep participation in mathematical exploration captivated these participants as they became aware of complexities. Students were compelled to find creative responses to the problems they uncovered in these open mathematical landscapes. As they chose how to approach and how to report on their work in these open spaces, their values were uncovered.

When students are captivated by the problems they uncover in their exploration, teachers might draw upon their students' findings to engage the rest of the class in dialogue about important mathematics. When students do not seem to be captivated by their exploration, teachers might consider what is distracting them. Uncovering that which interferes with their engagement might initiate further rich exploration or insightful discussion about mathematical investigation.

The experience of the students in my research might also lead teachers to consider how to expose their students to complexities inherent in their mathematics. Explicit dialogue with students regarding what is important and why it is important might uncover assumptions in their mathematics discourse and might also help students decide what is important to write and talk about when they communicate mathematically.

Students that can experience mathematics in these ways, I believe, will find their creativity supported. Investigative projects can be a valuable beginning for such experiences.

That Which I Had Hoped to See

Before describing the new mathematics education landscapes my research now compels me to investigate, I look again at my initial research questions. My first research objective was to examine within the classroom culture the shifts that can accompany project work. The experience for these teachers and students seemed to be that of sudden cross-cultural immersion. Although the people, the mathematical topics and the physical spaces were unchanged from normal classes, the mathematical place was unfamiliar.

The second objective was quite different: to use the cultural shifts within the classroom as a source of insight into possible implications of introducing pure mathematics investigative projects. I hoped to describe possible benefits of such projects by projecting the changes I observed into the participants' futures. Although I am still interested in the shaping effects of investigative projects, I found myself more interested in analyzing the investigative experience itself. I believe that, for people who focus on making healthier their present places of being, the future will take care of itself. Similarly, I believe that a rich mathematical experience that encourages sensitivity to complexity can only have positive effects on the participants' futures – the particular nature of these positive effects I do not need to know. In retrospect, the second objective – to extrapolate changes in classroom culture beyond the time and space of the project work – seems too grandiose for my brief participation with these two classes.

My research questions were closely related to my research objectives. My experience of these two classes sustained my interest in only the first three of the eight questions. Here I briefly respond to the last five of these questions, the ones that no longer captivate me.

My fourth question asked how the participant teachers would try to influence their students' interpretation of the unfamiliar parts of the instructions. The fifth was similar, with an interest in how the students would try to influence their teacher. Both of these questions expose my initial expectation that the participants would feel confident enough to influence the people around them. To my surprise, they were all disoriented. Teachers and students alike seemed to be too absorbed in their own feelings of indecision to pay much attention to influencing their neighbours.

Questions six and seven can be coupled to read as one question: In what ways would the unfamiliar parts of the instructions seem to free participant students and teachers to think and act in new ways? This pair of questions still interests me somewhat, but in a different way from the one I expected. In these investigative settings, the participants actually seemed bound, not freed, by the complexities of the mathematical places in which they were immersed. There is, however, a sense in which this binding loosed creativity. When simplistic responses to complex problems are closed off, participants are propelled into a zone of creativity.

With my final question I wondered what my research participants would say about the value of the unfamiliar parts of their instructions. They did not talk about the value of the unfamiliar parts. They talked about the familiar parts. When I asked students why such projects were good, they tended to talk about the value of working in groups rather than about the particularities of the unfamiliar open-ended questions. They were accustomed to group work from other subject areas – particularly in the humanities.

In retrospect, I feel as though I ought to have expected this. It seems normal to describe an experience in terms of familiar things. How can we use unfamiliar terms? My interpretation of these classroom experiences, for example, focuses on the aspects that are familiar to me. I compare the participants' reactions with my experiences of being immersed in a different culture.

The first three research questions, which focused on the unfamiliar aspects of the investigation tasks – the questions that continued to captivate me during my investigation and interpretation – are addressed through the discussion of common themes in Chapter 6. Each of the themes emerged from my experiences with immersion in foreign places and my sense of a parallel between my foreign experiences and the experiences of my research participants. This parallelism suggested to me the depth of their participation in the mathematical spaces opened up for them by the two investigative projects.

Tensions

Although I chose my research questions myself, my research project was similar to the open-ended projects I provided for my participant teachers and students. With my study, I have been drawn deeply, albeit for a short time, into an exploration of two pedagogical spaces and the many smaller places within those spaces. As with the students' brief immersions into mathematical territory, my participation in these classroom landscapes captivated me. I have used this experience as a reference point in making sense of other important events around me. With my ever-increasing captivation, I become more and more aware of the complexity of the places I was investigating. Because I did not want to be complacent and make simplistic interpretations, I was propelled into a zone of creativity – a zone in which I found a coherent but incomplete way of reporting. My participation in the lives of these students, their teachers and their mathematics was filled with the tensions that are characteristic of deeper relationships.

Just as I interpreted the tensions experienced by my participants as reflections of the depth and quality of their mathematical immersions, I feel affirmed by the tensions I sense in this presentation. Mason (1988) speaks of the potent source of energy we have at our disposal when we experience tension. Too often we are numbed to the point of inaction when we struggle within ourselves to find an appropriate position within a landscape of endless possibility.

Mr. Penner and Mrs. Foster struggled to find a good way of guiding their students in open mathematical landscapes, while their students struggled to find points of reference. They all experimented, to some extent, with various ways of positioning themselves, but, regrettably, most students were numbed into inaction when it came to reporting on their investigation-based findings. The little writing these students did on their posters did not reflect the depth of their mathematical thinking. Their problem, I believe, was that they sensed that their teachers would not value their good ideas. Indeed, the students themselves likely did not value their own good ideas.

Similarly, I struggled to find a way to present the results of my opened-up thinking such that it might become valuable to my readers. I struggled because my descriptions and comparisons would lay bare my values, and I feared that my exposed values would offend readers who hold different values. However, I did not want to be numbed into inaction like most of my participating students; I did not want to choose a "safe" path similar to the one described by Alrø and Skovsmose (1998).

I have already shared with a number of people the way of seeing that emerged from my research. Some of these people seemed offended by my apparently negative portrayal of tourists. Since most of the people with whom I have shared these ideas have been tourists at one time or another, it is understandable that they would consider themselves criticized by implication. Indeed, I have been a tourist and will probably be a tourist again. Is tourism so bad?

No. I believe tourism can be a positive way of experiencing a new place on the surface – as a recreational diversion and even as a potent source of growth and new-found understanding. Problems occur when tourists claim deep knowledge despite a truly shallow view of the complex places they describe. Because of physical, temporal and resource limitations, it is impossible to experience deep immersion in every complex community in the world. However, as we have more experiences of immersion in complex places new to us, we can develop an awareness that allows us to read complexity into our subsequent shallow glimpses of other places. When I was a tourist in Zimbabwe, for example, I was well aware of the shallowness of my experience and, because of this awareness, I was able to look for complexity even in my brief time there.

I suggest that it is the same for mathematical excursions. Problems occur when students are given whirlwind tours that provide simple looks at a smattering of "must-see" tour stops. If this kind of experience is the extent of their participation in mathematical spaces, they will be as frustrated as Greg was when confronted by the very real complexity that underlies these spaces. Such a mathematical experience also robs students of their potential awareness of connecting spaces between these must-see sights. By contrast, if students are invited or propelled into investigations of open mathematical landscapes, even periodically, they ought to be able to draw a richer understanding from subsequent brief and shallow stops.

Some Uncovered Problems

Although I do see potential for global tourism and mathematical tourism, I am still captivated by what I feel to be the greater potential of immersion experiences. I would like to see a mathematics classroom environment that resembles Jaworski's (1994) description of rich investigative settings, in which students direct their own mathematical exploration (p. 3).

Such an environment resembles my experiences with graduate studies. I have followed my interests and inclinations to set my own problem. And, I have enjoyed immersing myself for a little while in the problem. I see endless depth in such an approach because thoughtful responses to complex problems naturally expose new problems. Here I outline a few problems that have been exposed in my brief engagement with my primary research problem.

The Problem of Real Problems

Borasi (1992) compares mathematics problems to real-life problems. My understanding of a real problem is a situation where an unfulfilled need has no apparent resolution. This understanding begs questions about students' experiences with mathematical problems that consume them. What unfulfilled "need" does their immersion in such problems satisfy? Or, what unfulfilled "need" do students consciously or unconsciously hope to fulfill? Why are we (some of us) compelled to do mathematics? What kinds of pressing, unsolvable problems do Canadian children internalize?

My initial thoughts lean in the direction of seeing an underlying connection between all the problems in the world. As we begin to take a particular problem seriously, we cannot avoid the complexity inherent in it and we cannot avoid uncovering other problems. We can shelter ourselves from complexity by blanketing the links to other problems with limiting or denying assumptions, but when we begin to reconsider our assumptions the connected problems rear their heads once again.

The beginnings of an example might help here. I can only give the beginnings of an example, because the point of my suggestion is that there is no end to any examples of the interconnectedness of problems.

When Greg and his group were frustrated with "Parallel Division", they were actually engaged in what I would call "good mathematics". Unfortunately they did not value their own discourse. A question behind this issue might be: what is important in mathematical thinking and action? What is more important: a decisive answer that ignores complexity, or a slowly developed response that respects the complexity of the place? Stronger ways of posing this question might ask which approach is more ethical, more useful, more empowering, or more life-affirming?

This question could apply to any pursuit or interest I have in the world. When I respond to real problems in complex spaces, either in my home country or abroad, is it better to ignore complexities and act on understandably limited information gathered from a narrow perspective or is it better to spend my time pursuing the complexities and sharing my findings with other participants in the problem space? Probably the best approach would be characterized by an awareness of both approaches and by features of both.

Skovsmose (2000) suggests that the way we approach real-life problems is formatted by our mathematics education experience. While I believe there is truth in his assertion, I am also interested in how our culture's way of approaching its problems expresses itself in our mathematics pedagogy. Mathematics education informs our living outside the classroom at the same time as life outside the classroom drives the form of mathematics education.

I am beginning to feel like the roots of all problems are linked together – mathematical problems, dysfunctional relationship problems, distribution of resource problems. Yes, all problems.

Giving Problems

I find myself often reflecting on Kilpatrick's (1987) words, which I quote earlier: "One person cannot give a problem to another person" (p. 124). Many questions surface in my reflection. With each of these questions, I am thinking about both mathematical problems as well as not-necessarily-mathematical problems. Why would I want to give someone a problem anyway? Aren't problems a bad thing to have? Assuming that some problems might be good to have, what is a good problem? How can I help people to experience these good kinds of problems?

With mathematical problems, it seems that if a teacher cannot give a problem to students then the teacher can lead them into a problem-fraught mathematical space where it is likely that they will become captivated by a problem. Open-ended investigative tasks, such as the two considered in my research, can provide this kind of space. In such a space, however, there is potential for students to be distracted from a compelling problem.

I noticed that the students in my research were sometimes distracted by the written investigative tasks, and by their feeling that they needed to address every part in the tasks. For example, in Chapter 6, I describe how Terry, Brian and Shawn were captivated by their exploration until Terry noticed the instructions lying on his desk. They immediately stopped exploring to write hastily-constructed answers to the questions contained in their instructions.

How might such distractions be averted? I am beginning to experiment with giving tasks orally instead of on paper, with the hope that the problems might reside inside the explorers' thoughts instead of on paper. Problems on paper exist external to the participants' thinking with the additional authority of the printed word. I would expect that different students would imagine the central problem that arises out of an oral prompt in different ways. As students explore independently, they would cultivate fertile ground for a discussion of the connections between the mathematical problems that consume their attention.

Problematic Discourse

As I reflect again on the interconnectedness of the problems of mathematics discourse experienced by the students in this research and the not-necessarily-mathematical problems experienced by people entering foreign places, my attention is turned once again to my three-year experience in Swaziland. The most frustrating times for my family and myself were the times when we did not have the language and experience to describe the questions that lurked at the edges of our awareness. Our friendly neighbours described for us in painstaking detail things that seemed quite obvious to us. However, no one explained the things that truly mystified us. I realize now that this problem should come as no surprise, because it is difficult for us to see the assumptions that underpin our own traditional ways of living. Hence we often need a foreigner to help us see the problems in our culture.

This research experience has uncovered for me a realization that too many students experience similar frustrations every time they walk into a mathematics classroom. Every day they enter a foreign place where they are mere tourists, sheltered from the assumptions and other complexities that permeate their discourse. Many students recognize the shallowness of their experience, but without sufficient alternative experience, they either wonder how to break through the barriers separating them from the rich landscape or despair of the vacuous view of the world presented to them.

I suggest that those of us interested in the development of our society's young people need to explore ways in which students and teachers can become more aware of the idiosyncrasies and particularities of their mathematics classroom language and culture. Perhaps students will be enabled to understand and articulate their ideas and questions in both routine and new mathematical experiences. Because of the connections between the problems in their mathematics classrooms and problems in the rest of the real world, perhaps these students will be enabled to understand and articulate new ideas that emerge out of a sensitive approach to the complexity of their world.

References

Ainley, J. (1987). Telling questions. Mathematics Teaching, 118, 24-26.

Ainley, J. (1988). Perceptions of teachers' questioning styles. Paper presented at the Psychology of Mathematics Education Conference, Vezsprem, Hungary.

Ainley, J. (1999). Who are you today? Complementary and conflicting roles in school-based research. For the Learning of Mathematics, 19 (1), 39-47.

Alberta Learning (2000). Outcomes with assessment standards for Pure Mathematics 10 (interim report). Edmonton: [Author].

Alrø, H. and Skovsmose, O. (1996). On the right track. For the Learning of Mathematics, 16 (1), 2-8.

Alrø, H. and Skovsmose, O. (1998) That was not my intention! Communication in mathematics education. For the Learning of Mathematics, 18 (2), 42-51.

Andrews, P. (1987). Answers to telling questions. Mathematics Teaching, 119, 12-13.

Baker, K. (1986). The other side of the moon. Mathematics Teaching, 115, 48-49.

Ball, D. and Wilson, S. (1996). Integrity in teaching: Recognizing the fusion of the moral and intellectual. American Educational Research Journal, 33 (1), 155-192.

Bishop, A. (1994). Cultural conflicts in mathematics education: Developing a research agenda. For the Learning of Mathematics, 14 (2), 15-18.

Borasi, R. (1992). Learning mathematics through inquiry. Portsmouth: Heinemann.

Brown, L., Hewitt, D. and Mason, J. (1994). Ways of seeing. In M. Selinger (Ed.) Teaching mathematics (pp. 85-94). London: Routledge.

Brown, S. and Walter, M. (1990). The art of problem posing (2nd edition). Hillsdale, New Jersey: Lawrence Erlbaum.

Chazan, D. and Ball, D. (1999). Beyond being told not to tell. For the Learning of Mathematics, 19 (2), 2-10.

Confrey, J. (1995). A theory of intellectual development. For the Learning of Mathematics, 15 (2), 36-45.

Cornelius, M. (1985). Problems for a mathematical department. Mathematics Teaching, 110, 38-39.

Davis, B. (1997) Listening for differences: An evolving conception of mathematics teaching. Journal for Research in Mathematics Education, 28 (3), 355-376.

Delaney, K. (1986). Vestigating. Mathematics Teaching, 114, 16.

Demi (1990). The empty pot. New York: Henry Holt.

Department of Education and Science (DES) (1982). Mathematics counts: Report of the Committee of Inquiry into the Teaching and Learning of Mathematics in Schools under the chairmanship of W. H. Cockcroft. London: HMSO.

Edmonds, B. (1983). Innovation isn't easy. Mathematics Teaching, 103, 33-34.

Fennema, E., Franke, M., Carpenter, T. and Carey, D. (1993). Using children's mathematical knowledge in instruction. American Educational Research Journal, 30 (3), 555-583.

Fielker, D. (1982). Examinations: Assessment: Problems. Mathematics Teaching, 100, 46-53.

Fielker, D. (1983). Removing the shackles of Euclid 9: Syllabus. Mathematics Teaching, 104, 26-29.

Forman, E., McCormick, D. and Donato, R. (1998). Learning what counts as a mathematical explanation. Linguistics and Education, 9 (4), 313-319.

Frankham, D. (1983). Passages. Mathematics Teaching, 104, 15.

Gattegno, C. (1981). Children and mathematics: A new appraisal. Mathematics Teaching, 94, 5-7.

Gerofsky, S. (1996). A linguistic and narrative view of word problems in mathematics education. For the Learning of Mathematics, 16 (2), 36-45.

Gerofsky, S. (1999). The word problem as genre in mathematics education. (Doctoral dissertation, Simon Fraser University, 1999) Dissertation Abstracts International, AATNQ51864.

Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: Aims, rationales, techniques. For the Learning of Mathematics, 1 (3), 4-11.

Goldsmith, L. and Shifter, D. (1997). Understanding teachers in transition: characteristics of a model for the development of mathematics teaching. In E. Fennema and B. Nelson (Eds.). Mathematics teachers in transition (pp. 19-54). Mahwah, New Jersey: Lawrence Erlbaum.

Hewitt, D. (1983). Passages. Mathematics Teaching, 104, 14.

Hewitt, D. (1987). Memory. Mathematics Teaching, 118, 18-20.

Hewitt, D. (1992). Train Spotter's Paradise. Mathematics Teaching, 140, 6-8.

Hewitt, D. (1999). Arbitrary and necessary: Part 1 – A way of viewing the mathematics curriculum. For the Learning of Mathematics, 19 (3), 2-9.

Hewitt, D. (2001a). Arbitrary and necessary: Part 2 – Assisting memory. For the Learning of Mathematics, 21 (1), 44-51.

Hewitt, D. (2001b). Arbitrary and necessary: Part 3 – Educating awareness. For the Learning of Mathematics, 21 (2), 37-49.

Jaworski, B. (1985). A poster lesson. Mathematics Teaching, 113, 4-5.

Jaworski, B. (1994). Investigating mathematics teaching: A constructivist enquiry. London: Falmer.

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. Schoenfeld (Ed.). Cognitive science and mathematics education (pp. 123-147). Hillsdale, New Jersey: Lawrence Erlbaum.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. American Educational Research Journal, 27 (1), 29-63.

Lappan, G. (1997). Challenges of implementation: Supporting teachers. American Journal of Education, 106, (November 1997), 207-239

Lessing, D. (1992). African Laughter: Four visits to Zimbabwe. New York: Harper Collins.

Love, E. (1988). Evaluating mathematical activity. In D. Pimm (Ed.). Mathematics, teachers and children (pp. 249-262). London: Hodder and Stoughton.

Mason, J. (1988). Tensions. In D. Pimm (Ed.). Mathematics, teachers and children (pp. 164-169). London: Hodder and Stoughton.

Mason, J. and Muller, E. (2001). Where is the mathematics? Working group at the Canadian Mathematics Education Study Group, Edmonton, Alberta.

Mason, J. and Watson, A. (1998). Questions and prompts for mathematical thinking. Derby: Association of Teachers of Mathematics.

Mason, J., Burton, L. and Stacey, K. (1982). Thinking mathematically. London: Addison-Wesley.

McCabe, K. (2000). The case of the implementation of the Western Canadian Protocol in Mathematics in Alberta. (Masters thesis, University of Alberta, 2000) Dissertation Abstracts International, [number pending].

Morgan, C. (1998). Writing mathematically: The discourse of investigation. London: Falmer.

Mulholland, L. (1985). Quick quick slow. Mathematics Teaching, 113, 42-43.

NCTM (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: National Council of Teachers of Mathematics.

NCTM (2000). Principles and Standards in School Mathematics [Overview]. Reston, VA: National Council of Teachers of Mathematics.

Nemirovsky, R. (1996). A functional approach to algebra: two issues that emerge. In N. Bednarz, C. Kieran and L. Lee (Eds.). Approaches to algebra: Perspectives for research and teaching (pp. 295-342). Dordrecht: Kluwer.

Noss, R. (1983). Doing maths while learning Logo. Mathematics Teaching, 104, 5-10.

OED (2nd ed., 1989) (Eds. J. A. Simpson and E. S. C. Weiner). Oxford English Dictionary Online. Oxford University Press. [On-line] Available: http://oed.com

Pauls, K. (2001). The conflict between faith and duty: The story of Mennonite men who went to war. In Tapestry (September 23). Winnipeg: Canadian Broadcasting Corporation.

Rowland, T. (2000). The pragmatics of mathematics education: Vagueness in mathematical discourse. London: Falmer.

Schoenfeld, A. (1988). Problem solving in contexts. In R. Charles and E. Silver (Eds.). The teaching and assessing of mathematical problem solving (pp. 82-92). Hillsdale, New Jersey: Lawrence Erlbaum.

Shuller, N. (1983). Working with images. Mathematics Teaching, 104, 38-41.

Skovsmose, O. (2000). Aporism and critical mathematics education. For the Learning of Mathematics, 20 (1), 2-8.

Skovsmose, O. and Nielsen, L. (1996). Critical mathematics education. In Bishop, A. *et al*. (Eds.). International handbook of mathematics education (pp. 1257-1288). Dordrecht: Kluwer.

Smith, J. (1986). Questioning questioning. Mathematics Teaching, 115, 47.

von Glasersfeld, E. (1995). Piaget's constructivist theory of knowing. In Radical Constructivism: A way of knowing and learning (pp. 53-75). London: Falmer.

Wagner, D. (1998). Letter from Swaziland: all is not as it first appears. The Alberta Teachers Association News, 32 (17), 5.

Watson, A. (1986). Opening up. Mathematics Teaching, 115, 16-18.

Wheeler, D. (1984). Gatherings. Mathematics Teaching, 106, 24-25.

Wheeler, D. (1988). Investigating Investigations. In D. Pimm (Ed.). Mathematics, teachers and children (pp. 303-305). London: Hodder and Stoughton.

Wilensky, U. (1991). Abstract meditations on the concrete and concrete implications for mathematics education. In I. Harel and S. Papert (Eds.). Constructionism (pp. 193-203). Norwood, New Jersey: Ablex.

Williams, S. and Baxter, J. (1996). Dilemmas of discourse-oriented teaching in one middle school mathematics classroom. The Elementary School Journal, 97 (1), 21-38.

Appendix 1: Scoring Rubrics

Generic Scoring Rubric for Open Exploration (2 pages)

Mark	Creativity	Mathematical Concepts and Processes
5	• creative approach which *inspires further thought* and exploration • mathematical techniques and ideas are *beyond expectations.*	• *complete understanding* of the mathematical concepts and processes used • *all important elements* of the task are completed
4	• *creative* approach to bringing *ideas from outside* the course's specific content • mathematical techniques and ideas are *beyond expectations.*	• *good understanding* of the mathematical concepts and processes used • *most* of the important *elements* of the task are completed.
3	• brings *ideas from outside* the course's specific content. • mathematical techniques and ideas *meet expectations.*	• an *understanding of most* mathematical concepts and processes used • *some* of the important *elements* of the task are completed
2	• an *ineffectual attempt* at bringing ideas from outside the course's specific content. • mathematical techniques and ideas are *below expectations.*	• *some understanding* of mathematical concepts and processes used • *only a few elements* of the task are completed
1	• no evidence of thought beyond the most *obvious approaches* to the problem. • mathematical techniques and ideas are *inappropriate* for this level of study.	• *very limited understanding* of mathematical concepts and processes used • *only superficial elements* of the task are completed
0	• *insufficient work* for judging the engagement of thought.	• *no understanding* of mathematical concepts or processes is evident.

___ x 2 → $\frac{\_\_}{10}$　　　　___ x 2 → $\frac{\_\_}{10}$

Note: This scoring rubric was used by both Mr. Penner and Mrs. Foster. However, for the second project, "Parallel Division", Mr. Penner used this rubric without the "Creativity" column.

Mark	Communication	Presentation
5	• *clear and concise* communication of ideas with *supporting graphics*. • demonstrates *higher-level mathematical thinking* (e.g. conjectures, generalizations, examples, counterexamples)	• *clear*, organized and informative • clear and accurate graphics • *dynamic* (using diverse methods of presentation)
4	• *clear* communication of ideas with *supporting graphics.* • clear evidence of *mathematical thinking* (e.g. comparisons, conjectures)	
3	• *relatively clear* communication of ideas but lacking supporting detail. • *some evidence* of mathematical thinking.	• *reasonably clear* and organized • graphing may have *minor flaws* in accuracy and clarity. • minimal evidence of creativity in method of presentation.
2	• communication of ideas *lacks clarity* **or** *lacks graphics*. • fails to demonstrate coherent mathematical thinking (e.g. ineffective analysis, unclear argument, inappropriate interpretation.)	
1	• ideas are *superficially communicated* • explanations and justifications may be *convoluted or illogical.*	• *poorly* organized and superficial • graphs and diagrams do not follow mathematical conventions • inappropriate for intended audience
0	• no ideas are communicated or communication is inappropriate for the topic.	• no evidence of organization • lacks clarity • poor

___ x 2 → $\frac{\ }{10}$

___ x 1 → $\frac{\ }{5}$

Appendix 2: Letters of Consent

Dear Student and Parent(s)/Guardian(s),

Since your class will be engaging in project work this term I would like to ask you for your participation in my study of project work in grade 10 mathematics classrooms. Teacher and student responses to the project instructions will be the focus of my investigation of changes within the mathematics classroom. Both teacher and student views on differences between project instructions and more typical classroom instructions will be sources of insight into possible implications of project work. Findings in this study will be part of my master's thesis and may contribute to articles and presentations for teachers and teacher educators.

If you agree to participate in the study you may be selected to participate in two fifteen-minute audio-taped group interviews. Classroom interactions in five classes will be recorded in the form of written notes, audio-tape and videotape. The projects that you complete in the two classes devoted to project work will also be photocopied. You will be given the chance to confirm or withdraw your involvement at the beginning of each session in which I will be conducting research. Throughout my research I will endeavour to be open with you about my intentions, and to avoid deception.

If you agree to participate in this study anonymity will be maintained through the use of a pseudonym. The name of your school and district will not be identified. Only the researcher and his faculty advisor will know your identity. Data collected during this study will be secured in the researcher's office and any identifying information will be removed. Transcripts from your interview will be provided to you prior to research analysis in order for you to be able to confirm accuracy.

Participants will be given the chance to confirm or withdraw their involvement at the beginning of each session in which I will be conducting research. Should the teacher choose to opt out or withdraw at any time I will not conduct research in this class. Students who choose to opt out or withdraw at any time will not be interviewed. Depending on the preferences of the student, the teacher will find a place for the student to work either independently or outside of the range of tape recorders and video cameras. I will make no notes about such students. Students will not be able to opt out from the project work because the projects are part of the regular class activity and already part of their course outlines.

Your participation is voluntary and you are free to stop participating at any time, or to decline to answer any specific questions(s). Any questions regarding the research can be directed to David Wagner (email at davewag@oanet.com or phone at 492-0148) or David Pimm (email at david.pimm@ualberta.ca or phone at 492-0150). The results of this research will also be provided upon email request to David Wagner.

Your participation in this study is greatly appreciated. By reviewing and signing the attached form, you are agreeing to informed consent of this study. Please return the form to your classroom teacher.

Thank you kindly,

David Wagner

Dear Teacher,

Since your class will be engaging in project work this term I would like to ask you for your participation in my study of project work in grade 10 mathematics classrooms. Teacher and student responses to the project instructions will be the focus of my investigation of changes within the mathematics classroom. Both teacher and student views on differences between project instructions and more typical classroom instructions will be sources of insight into possible implications of project work. Findings in this study will be part of my master's thesis and may contribute to articles and presentations for teachers and teacher educators.

If you agree to participate in the study you will participate in two thirty-minute audio-taped interviews. Classroom interactions in five classes will be recorded in the form of written notes, audio-tape and videotape. Your grading and evaluation comments regarding projects that are completed in the two classes devoted to project work will also be photocopied. You will be given the chance to confirm or withdraw your involvement at the beginning of each session in which I will be conducting research. Throughout my research I will endeavour to be open with you about my intentions, and to avoid deception.

If you agree to participate in this study anonymity will be maintained through the use of a pseudonym. The name of your school and district will not be identified. Only the researcher and his faculty advisor will know your identity. Data collected during this study will be secured in the researcher's office and any identifying information will be removed. Transcripts from your interview will be provided to you prior to research analysis in order for you to be able to confirm accuracy.

Participants will be given the chance to confirm or withdraw their involvement at the beginning of each session in which I will be conducting research. Should the teacher choose to opt out or withdraw at any time I will not conduct research in this class. Students who choose to opt out or withdraw at any time will not be interviewed. Depending on the preferences of the student, the teacher will find a place for the student to work either independently or outside of the range of tape recorders and video cameras. I will make no notes about such students. Students will not be able to opt out from the project work because the projects are part of the regular class activity and already part of their course outlines.

Your participation is voluntary and you are free to stop participating at any time, or to decline to answer any specific questions(s). Any questions regarding the research can be directed to David Wagner (email at davewag@oanet.com or phone at 492-0148) or David Pimm (email at david.pimm@ualberta.ca or phone at 492-0150). The results of this research will also be provided upon email request to David Wagner.

Your participation in this study is greatly appreciated. By reviewing and signing the attached form, you are agreeing to informed consent of this study.

Thank you kindly,

David Wagner

University of Alberta
Research Consent Form

I, ____, hereby consent
(print name of parent/legal guardian or independent student)

for ____ to
(print name of student)

- participate in two group interviews (15 minutes each)
- be audio-taped in classroom interaction (5 occasions)
- be videotaped in classroom interaction (5 occasions)
- allow the analysis of two classroom projects

by David Wagner

I understand that:

- my child may withdraw from the research at any time without penalty
- all information gathered will be treated confidentially and discussed only with your supervisor
- any information that identifies my child will be destroyed upon completion of this research
- my child will not be identifiable in any documents resulting from this research

I also understand that the results of this research will be used only in the following:

- master's thesis
- presentations and written articles for other educators

________________________________ or ________________________________
signature of parent/legal guardian signature of student if 18 years or older

Date signed: ________________________

For further information concerning the completion of this form, please contact David Wagner or David Pimm at 492-0148 or at 341 Education South, Faculty of Education, University of Alberta
Edmonton, Alberta, T6G 2G5.

University of Alberta
Research Consent Form

I, ____, hereby consent to
(print name of teacher)

- participate in two group interviews (30 minutes each)
- be audio-taped in classroom interaction (5 occasions)
- be videotaped in classroom interaction (5 occasions)
- allow the analysis of my comments and grading on two sets of student projects

by ____David Wagner__________

I understand that:

- I may withdraw from the research at any time without penalty
- all information gathered will be treated confidentially and discussed only with your supervisor
- any information that identifies me will be destroyed upon completion of this research
- I will not be identifiable in any documents resulting from this research

I also understand that the results of this research will be used only in the following:

- master's thesis
- presentations and written articles for other educators

____
signature of teacher

Date signed: __________________________

For further information concerning the completion of this form, please contact David Wagner or David Pimm at 492-0148 or at 341 Education South, Faculty of Education,
University of Alberta
Edmonton, Alberta, T6G 2G5.